Chapter 12. **Partial derivatives.**

Section 12.6 **Directional derivatives and the gradient vector.**

Let $z = f(x, y)$. We wish to find the rate of change of z at (x_0, y_0) in the direction of an arbitrary unit vector $\vec{u} =< a, b >$.

To do this we consider the surface S with equation $z = f(x, y)$ and we let $z_0 = f(x_0, y_0)$. Then the point $P(x_0, y_0, z_0)$ lies on S. The vertical plane that passes through P in the direction $\vec{u}$ intersects S in a curve C. The slope of the tangent line T to C at P is the rate of change of change of z in the direction of $\vec{u}$.

Let $Q(x, y, z)$ be another point on C. If $P'(x_0, y_0, 0)$ and $Q'(x, y, 0)$ are projections of P and Q on the xy-plane, then the vector $\overrightarrow{P'Q'} =< x - x_0, y - y_0, 0 >$ is parallel to $\vec{u}$ and so

$$\overrightarrow{P'Q'} = h\vec{u} =< ha, hb >$$

for some scalar h. Therefore

$$x - x_0 = ha \quad y - y_0 = hb$$

and

$$\frac{\Delta z}{h} = \frac{z - z_0}{h} = \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

If we take the limit as $h \to 0$, we obtain the rate of change of z in the direction of $\vec{u}$, which is called the directional derivative of f in the direction of $\vec{u}$.

Definition. The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\vec{u} =< a, b >$ is

$$D_{\vec{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{z - z_0}{h} = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

Theorem. If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\vec{u} =< a, b >$ and

$$D_{\vec{u}} f(x, y) = \frac{\partial f}{\partial x}(x, y)a + \frac{\partial f}{\partial y}(x, y)b$$

If the unit vector $\vec{u}$ makes an angle θ with the positive x-axis, then

$$D_{\vec{u}} f(x, y) = \frac{\partial f}{\partial x}(x, y)\cos\theta + \frac{\partial f}{\partial y}(x, y)\sin\theta$$

Example 1. Find the directional derivative of the function $f(x, y) = y^x$ at the point $(1, 2)$ in the direction of the unit vector $\vec{u}$ given by angle $\theta = -2\pi/3$.

Definition. If f is a function of two variables x and y, then the **gradient** of f is defined by

$$\text{grad}(f) = \nabla f(x, y) = \left\langle \frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial y}(x, y) \right\rangle.$$

Then

$$D_{\vec{u}} f(x, y) = \nabla f(x, y) \cdot \vec{u}.$$

Example 2. Find the directional derivative of the function $f(x, y) = xe^{xy}$ at the point $(-3, 0)$ in the direction of the vector $\vec{v} = 2\vec{\imath} + 3\vec{\jmath}$.

For a function of three variables $w = f(x, y, z)$ the **gradient vector** is

$$\text{grad}(f) = \nabla f(x, y, z) = \left\langle \frac{\partial f}{\partial x}(x, y, z), \frac{\partial f}{\partial y}(x, y, z), \frac{\partial f}{\partial z}(x, y, z) \right\rangle$$

and

$$D_{\vec{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \vec{u}.$$

Theorem. Suppose f is a differentiable function of two or three variables and $\vec{x} =< x, y >$ if f is a function of two variables $\vec{x} =< x, y, z >$ if f is a function of three variables. The maximum value of the directional derivative $D_{\vec{u}} f(\vec{x})$ is $|\nabla f(\vec{x})|$ and it occurs when $\vec{u}$ has the same direction as the gradient vector $\nabla f(\vec{x})$.

Example 3. Suppose that over a certain region of space the electrical potential V is given by $V(x, y, z) = 5x^2 - 3xy + xyz$.

1. Find the rate of change of the potential at $P(3, 4, 5)$ in the direction of the vector $\vec{v} =< 1, 1, -1 >$.

2. In which direction does V change most rapidly at P?

3. What is the maximum rate of change at P?

Tangent planes to level surfaces.

Suppose S is a surface with equation $F(x, y, z) = k$, that is, it is a level surface of the function $w = F(x, y, z)$, and let $P(x_0, y_0, z_0)$ be a point on S.

We define the **tangent plane to the level surface** $F(x, y, z) = k$ **at** $P(x_0, y_0, z_0)$ as the plane that passes through P and has normal vector $\nabla F(x_0, y_0, z_0)$ and its equation is

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

The **normal line** to S at P is the line passing through P and perpendicular to the tangent plane (its direction is given by the gradient vector $\nabla F(x_0, y_0, z_0)$). Its symmetric equations are

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

If the equation of a surface S is of the form $z = f(x, y)$, we can rewrite

$$F(x, y, z) = f(x, y) - z = 0$$

and regard S as a level surface of F with $k = 0$. Then

$$F_x(x_0, y_0, z_0) = f_x(x_0, y_0), \quad F_y(x_0, y_0, z_0) = f_y(x_0, y_0), \quad F_z(x_0, y_0, z_0) = -1$$

so the equation of the tangent plane to S at (x_0, y_0) is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$$

and the equation of the normal line to S at (x_0, y_0) is

$$\frac{x - x_0}{f_x(x_0, y_0)} = \frac{y - y_0}{f_y(x_0, y_0)} = \frac{z - z_0}{-1}$$

Example 4. Find equations of the tangent plane and the normal line to the surface $x^2 - 2y^2 - 3z^2 + xyz = 4$ at the point $(3, -2, -1)$.

Example 5. Find parametric equations for the tangent line to the curve of intersection of the paraboloid $z = x^2 + y^2$ and the ellipsoid $4x^2 + y^2 + z^2 = 9$ at the point $(-1, 1, 2)$.