Chapter 12. **Partial derivatives.**
Section 12.8 **Lagrange multipliers.**

Problem. Find the extreme values of the function $w = f(x, y, z)$ subject to the constraint $g(x, y, z) = k$.

To find the maximum and minimum values of $f(x, y, z)$ **subject to constraint** $g(x, y, z) = k$ (assuming that these extreme values exist):

1. Find all values of x, y, z, and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

and

$$g(x, y, z) = k$$

2. Evaluate f at all the points (x, y, z) that arise from step 1. The largest of these values is the maximum value of f; the smallest is the minimum value of f.

The number λ is called a **Lagrange multiplier**. The procedure described above is called the **method of Lagrange multipliers**.

If we rewrite the vector equation $\nabla f = \lambda \nabla g$ in terms of its components, then the equation in step 1 become

$$f_x = \lambda g_x, \qquad f_y = \lambda g_y, \qquad f_z = \lambda g_z, \qquad g(x, y, z) = k$$

Example 1. Use Lagrange multipliers to find the maximum and minimum values of the function $f(x, y) = x^2 + y^2$ subject to the constraint $x^4 + y^4 = 1$.

Suppose now we want to find the maximum and minimum values of $f(x, y, z)$ subject to two constrains (side conditions) of the form $g(x, y, z) = k$ and $h(x, y, z) = c$. There are numbers λ and μ (called Lagrange multipliers) such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z)$$

In this case Lagrange's method is to look for extreme values by solving five equations

$$\begin{aligned} &f_x = \lambda g_x + \mu h_x \\ &f_y = \lambda g_y + \mu h_y \\ &f_z = \lambda g_z + \mu h_z \\ &g(x, y, z) = k \\ &h(x, y, z) = c \end{aligned}$$

in unknowns x, y, z, λ, and μ.

Example 2. The plane $x + y + 2z = 2$ intersects the paraboloid $z = x^2 + y^2$ in an ellipse. Find the points of this ellipse that are nearest to and farthest from the origin.