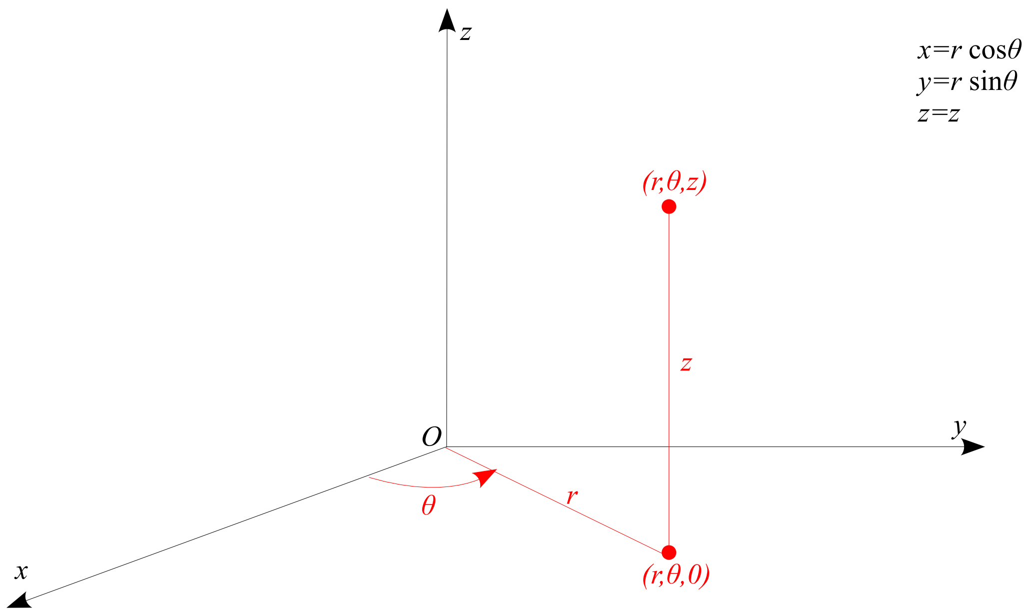
Chapter 13. **Multiple integrals.**

Section 13.10 **Triple integrals in cylindrical and spherical coordinates.**

Cylindrical coordinate system:



Suppose that E is a type 1 region whose projection D on the xy-plane is described in polar coordinates

$$E = \{(x, y, z) | (x, y) \in D, \varphi_1(x, y) \le z \le \varphi_1(x, y)\}$$

$$D = \{(r, \theta) | \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$$

Then

$$\iiint\limits_E f(x, y, z) dV = \iint\limits_D \left[\int_{\varphi_1(x,y)}^{\varphi_2(x,y)} f(x, y, z) dz \right] dA$$

If we switch to cylindrical coordinates

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z, \quad dA = r\,dr\,d\theta$$

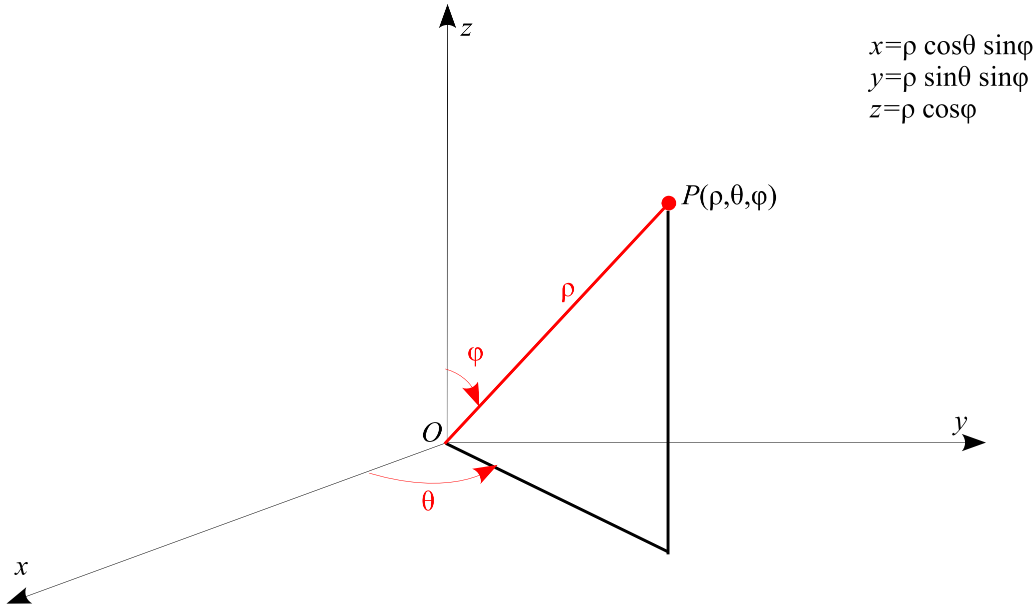
we will get

$$\iiint\limits_E f(x, y, z) dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{\varphi_1(r\cos\theta, r\sin\theta)}^{\varphi_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\,dz\,dr\,d\theta$$

Example 1. Sketch the solid whose volume is given by $\int_0^{2\pi} \int_0^2 \int_0^{4-r^2} r\, dz\, dr\, d\theta$.

Example 2. Evaluate $\iiint_E y dV$ where E is the solid that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$, above the xy-plane, and below the plane $z = x + 2$.

Spherical coordinate system:



In this coordinate system the analog of rectangular box is a **spherical wedge**

$$E = \{(\rho, \theta\varphi) | a \le \rho \le b, \alpha \le \theta \le \beta, \gamma \le \varphi \le \delta\}$$

where $a \ge 0$, $\beta - \alpha \le 2\pi$, and $\delta - \gamma \le \pi$.

To integrate over such region we consider a spherical partition P of E into smaller spherical wedges E_{ijk} by means of spheres $\rho = \rho_i$, half-planes $\theta = \theta_j$ and half-cones $\varphi = \varphi_k$. The norm of P, $\|P\|$, is the length of the longest diagonal of these wedges. If $\|P\|$ is small, then E_{ijk} is approximately a rectangular box with dimensions $\Delta\rho_i$, $\rho_i \Delta\varphi_k$ (arc of a circle with radius ρ_i, angle $\Delta\varphi_k$), and $\rho_i \sin\varphi_k \Delta\theta_j$ (arc of a circle with radius $\rho_i \sin\varphi_k$, angle $\Delta\theta_j$).

$$V(E_{ijk}) = \Delta V_{ijk} \approx \rho_i^2 \sin\varphi_j \Delta\rho_i \Delta\theta_j \Delta\varphi_k$$

Let $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ be the rectangular coordinates of the point in E_{ijk}. Then there are exist ρ^*_{ijk}, θ^*_{ijk}, and φ^*_{ijk} such that

$$x^*_{ijk} = \rho^*_{ijk} \cos\theta^*_{ijk} \sin\varphi^*_{ijk}, \qquad y^*_{ijk} = \rho^*_{ijk} \sin\theta^*_{ijk} \sin\varphi^*_{ijk}, \qquad z^*_{ijk} = \rho^*_{ijk} \cos\varphi^*_{ijk}$$

Then

$$\iiint\limits_E f(x,y,z)dV = \lim_{\|P\|\to 0} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\Delta V_{ijk} =$$

$$\lim_{\|P\|\to 0} f(\rho^*_{ijk}\cos\theta^*_{ijk}\sin\varphi^*_{ijk},\ \rho^*_{ijk}\sin\theta^*_{ijk}\sin\varphi^*_{ijk},\ \rho^*_{ijk}\cos\varphi^*_{ijk})[\rho^*_{ijk}]^2\sin\varphi^*_{ijk}\,\Delta\rho_i\,\Delta\theta_j\,\Delta\varphi_k =$$

Thus,

$$\iiint\limits_E f(x,y,z)dV = \int_\gamma^\delta\int_\alpha^\beta\int_a^b f(\rho\cos\theta\sin\varphi, \rho\sin\theta\sin\varphi, \rho\cos\varphi)\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$$

where $E = \{(\rho, \theta\varphi) | a \le \rho \le b, \alpha \le \theta \le \beta, \gamma \le \varphi \le \delta\}$

This formula can be extended to include more general spherical regions such as

$$E = \{(\rho, \theta\varphi) | \alpha \le \theta \le \beta, \gamma \le \varphi \le \delta\,, g_1(\theta,\varphi) \le \rho \le g_2(\theta,\varphi)\}$$

In this case

$$\iiint\limits_E f(x,y,z)dV = \int_\gamma^\delta\int_\alpha^\beta\int_{g_1(\theta,\varphi)}^{g_2(\theta,\varphi)} f(\rho\cos\theta\sin\varphi, \rho\sin\theta\sin\varphi, \rho\cos\varphi)\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$$

Example 3. Find the volume of the solid E that lies above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$.