Chapter 13. **Multiple integrals.**
Section 13.2 **Iterated integrals.**

Suppose f is a function of two variables that is integrable over the rectangle $R = [a, b] \times [c, d]$.

We use notation $\int\limits_c^d f(x, y)\ dy$ to mean that x is held fixed and $f(x, y)$ is integrated with respect to y from $y = c$ to $y = d$. This procedure is called **partial integration with respect to** y.

$$A(x) = \int\limits_c^d f(x, y)\ dy$$

$$\int\limits_a^b A(x)\ dx = \int\limits_a^b \left[\int\limits_c^d f(x, y)\ dy\right] dx$$

The integral $\int\limits_a^b \left[\int\limits_c^d f(x, y)\ dy\right] dx$ is called an **iterated integral**. Thus,

$$\int\limits_a^b \int\limits_c^d f(x, y) dy dx = \int\limits_a^b \left[\int\limits_c^d f(x, y)\ dy\right] dx$$

means that we first integrate with respect to y from c to d and then with respect to x from a to b.

Similarly, the iterated integral

$$\int\limits_c^d \int\limits_a^b f(x, y) dx dy = \int\limits_c^d \left[\int\limits_a^b f(x, y)\ dx\right] dy$$

means that we first integrate with respect to x from a to b and then with respect to y from c to d.

Example 1. Evaluate the iterated integrals:

1. $\int\limits_0^3 \int\limits_0^1 \sqrt{x + y}\ dx dy$

2. $\int_0^1 \int_0^1 \frac{xy}{\sqrt{x^2+y^2+1}} dydx$

Fubini's Theorem. If f is continuous on the rectangle $R = [a, b] \times [c, d]$, then

$$\iint_R f(x,y)\ dA = \int_c^d \int_a^b f(x,y) dxdy = \int_a^b \int_c^d f(x,y) dydx$$

Example 2. Calculate the double integral

$$\iint_R \left(xy^2 + \frac{y}{x}\right) dA,$$

where $R = \{(x,y)|2 \leq x \leq 3, -1 \leq y \leq 0\}$.

Example 3. Find the volume of the solid lying under the elliptic paraboloid $\frac{x^2}{4} + \frac{y^2}{9} + z = 1$ and above the rectangle $R = [-1, 1] \times [-2, 2]$.