Chapter 13. **Multiple integrals.**
Section 13.5 **Double integrals in polar coordinates.**

We want to evaluate

$$\iint\limits_R f(x,y)DA,$$

where R is a **polar rectangle**

$$R = \{(r,\theta)|a \le r \le b, \alpha \le \theta \le \beta\}$$

We start with a partition of [a,b] into m subintervals

$$a = r_0 < r_1 < r_2 < ... < r_m = b$$

and a partition of $[\alpha, \beta]$ into n subintervals

$$\alpha = \theta_0 < \theta_1 < \theta_2 < ... < \theta_n = \beta$$

Then the circles $r = r_i$ and the rays $\theta = \theta_j$ determine a **polar partition** P of R into the small polar rectangles. The norm $\|P\|$ of the polar partition is the length of the longest diagonal of the polar subrectangles.

The "center" of the polar subrectangle

$$R_{ij} = \{(r,\theta)|r_{i-1} \le r \le r_i, \theta_{j-1} \le \theta \le \theta_j\}$$

has polar coordinates

$$r_i^* = \frac{1}{2}(r_i + r_{i-1}) \qquad \theta_j^* = \frac{1}{2}(\theta_j + \theta_{j-1})$$

The area of the rectangle R_{ij} is

$$A(R_{ij}) = \Delta A_{ij} = \frac{1}{2}(r_i^2 - r_{i-1}^2)\Delta\theta_j = \frac{1}{2}(r_i + r_{i-1})(r_i - r_{i-1})\Delta\theta_j = r_i^*\Delta r_i\Delta\theta_j$$

where $\Delta\theta_j = \theta_j - \theta_{j-1}$, $\Delta r_i = r_i - r_{i-1}$.

The Cartesian coordinates of the center of R_{ij} are

$$(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)$$

so a Riemann sum is

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)\Delta A_{ij} = \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*) r_i^* \Delta r_i \Delta\theta_j$$

Therefore, we have

$$\iint_R f(x,y)dA = \lim_{\|P\|\to 0}\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*) r_i^* \Delta r_i \Delta\theta_j = \iint_R f(r\cos\theta, r\sin\theta) r\ dr\ d\theta =$$

$$\int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r\ dr\ d\theta$$

Change to polar coordinates in a double integral. If f is continuous on a polar rectangle R given by $0 \le a \le r \le b$, $\alpha \le \theta \le \beta$, where $0 \le \alpha - \beta \le 2\pi$, then

$$\iint_R f(x,y)dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r\ dr\ d\theta$$

Example 1. Evaluate the integral

$$\iint_R xy dA$$

where R is the region in the first quadrant that lies between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 25$.

If f is continuous on a polar region of the form

$$D = \{(r, \theta) | \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\},$$

then $$\iint\limits_R f(x, y)dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)r\ drd\theta$$

Example 2. Evaluate the integral

$$\iint\limits_D xdA$$

where R is the region in the first quadrant that lies between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 2x$.

Example 3. Use a double integral to find the area of the region enclosed by the lemniscate $r^2 = 4\cos 2\theta$.

Example 4. Use a double integral to find the area of the region inside the circle $r = 3\cos\theta$ and outside the cardioid $r = 1 + \cos\theta$.

Example 5. Use polar coordinates to find the volume above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$.