Chapter 13. **Multiple integrals.**
Section 13.8 **Triple integrals.**

We want to define the triple integrals for functions of three variables.
Let f is defined on a rectangular box

$$B = \{(x, y, z) | a \le x \le b, c \le y \le d, r \le z \le s\} = [a, b] \times [c, d] \times [r, s]$$

We partition the intervals $[a, b]$, $[c, d]$, and $[r, s]$ as follows:

$$a = x_0 < x_1 < ... < x_m = l$$

$$c = y_0 < y_1 < ... < y_n = m$$

$$r = z_0 < z_1 < ... < z_k = n$$

The planes through these partition points parallel to coordinate planes divide the box B into lmn sub-boxes

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$$

The volume of B_{ijk} is

$$\Delta V_{ijk} = \Delta x_i \Delta y_j \Delta z_k$$

where $\Delta x_i = x_i - x_{i-1}$, $\Delta y_j = y_j - y_{j-1}$, and $\Delta z_k = z_k - z_{k-1}$.

Then we form the **triple Riemann sum**

$$\sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \Delta V_{ijk}$$

where $(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \in B_{ijk}$. We define the **norm** $\|P\|$ of the partition P to be the length of the longest diagonal of all the boxes B_{ijk}.

Definition. The **triple integral** of f over the box B is

$$\iiint_B f(x, y, z) dV = \lim_{\|P\| \to 0} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \Delta V_{ijk}$$

if this limit exists.

Fubini's Theorem for triple integrals. If f is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint_B f(x, y, z) dV = \int_r^s \int_c^d \int_a^b f(x, y, z) dx dy dz$$

There are five other possible orders in which we can integrate.

Example 1. Evaluate the integral $\iiint_E (x^2 + yz)dV$, where

$$E = \{(x, y, z)|0 \le x \le 2,\ -3 \le y \le 0,\ -1 \le z \le 1\}$$

Now we define the **triple integral over a general bounded region** E in three-dimensional space.

A solid region E is said to be of **type 1** if it lies between the graphs of two continuous functions of x and y, that is,

$$E = \{(x, y, z)|(x, y) \in D, \varphi_1(x, y) \le z \le \varphi_2(x, y)\}$$

where D is the projection of E onto the xy-plane.

Then

$$\iiint\limits_E f(x, y, z)dV = \iint\limits_D \left[\int_{\varphi_1(x,y)}^{\varphi_2(x,y)} f(x, y, z)dz\right] dA$$

Example 2. Evaluate $\iiint\limits_E x dV$, where E is bounded by planes $x = 0$, $y = 0$, $z = 0$, and $3x + 2y + z = 6$.

A solid region E is of **type 2** if it is of the form

$$E = \{(x, y, z) | (y, z) \in D, \psi_1(y, z) \leq x \leq \psi_2(y, z)\}$$

where D is the projection of E onto the yz-plane.

Then

$$\iiint\limits_E f(x, y, z) dV = \iint\limits_D \left[\int_{\psi_1(y,z)}^{\psi_2(y,z)} f(x, y, z) dx \right] dA$$

A solid region E is of **type 3** if it is of the form

$$E = \{(x, y, z) | (x, z) \in D, \chi_1(x, z) \leq y \leq \chi_2(x, z)\}$$

where D is the projection of E onto the xz-plane.

Then

$$\iiint\limits_E f(x, y, z) dV = \iint\limits_D \left[\int_{\chi_1(x,z)}^{\chi_2(x,z)} f(x, y, z) dy \right] dA$$

Applications of triple integrals.

$$V(E) = \iiint\limits_E dV$$

Example 3. Find the volume of the solid bounded by the elliptic cylinder $4x^2 + z^2 = 4$ and the planes $y = 0$ and $y = z + 2$.

If the density function of a solid object that occupies the region E is $\rho(x, y, z)$ in units of mass per unit volume, at any given point (x, y, z), then its **mass** is

$$m = \iiint_E \rho(x, y, z) dV$$

and its **moments** about the three coordinate planes are

$$M_{yz} = \iiint_E x\rho(x, y, z) dV, \qquad M_{xz} = \iiint_E y\rho(x, y, z) dV, \qquad M_{xy} = \iiint_E z\rho(x, y, z) dV$$

The **center of mass** is located at the point $(\bar{x}, \bar{y}, \bar{z})$ where

$$\bar{x} = \frac{M_{yz}}{m} = \frac{\iiint_E x\rho(x, y, z) dV}{\iiint_E \rho(x, y, z) dV}, \qquad \bar{y} = \frac{M_{xz}}{m} = \frac{\iiint_E y\rho(x, y, z) dV}{\iiint_E \rho(x, y, z) dV}, \qquad \bar{z} = \frac{M_{xy}}{m} = \frac{\iiint_E z\rho(x, y, z) dV}{\iiint_E \rho(x, y, z) dV}$$

Example 4. Find the center of mass of a solid E with density $\rho(x, y, z) = x$, where E is bounded by the paraboloid $x = 4y^2 + 4z^2$ and the plane $x = 4$.