Chapter 14. **Vector calculus.**
Section 14.2 **Line integrals.**

Let C be a smooth plane curve with parametric equations

$$x = x(t), \qquad y = y(t), \qquad a \le t \le b$$

A partition of the parameter interval $[a, b]$ by points t_i with

$$a = t_0 < t_1 < ... < t_n = b$$

determine a partition P of the curve by points $P_i(x_i, y_i)$, where $x_i = x(t_i)$, $y_i = y(t_i)$, $z_i = z(t_i)$. Points P_i divide C into n subarcs with length Δs_1, Δs_2,...,Δs_n. The **norm** $\|P\|$ of the partition is the longest of these lengths. We choose any point $P_i^*(x_i^*, y_i^*)$ in the ith subarc.

Definition. If f is defined on a smooth curve C given by

$$x = x(t), \qquad y = y(t), \qquad a \le t \le b$$

then the **line integral of** f **along** C **with respect to arc length is**

$$\int_C f(x, y)ds = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$$

if this limit exists.

Since

$$ds = \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2} dt$$

then

$$\int_C f(x, y)ds = \int_a^b f(x(t), y(t))\sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2} dt$$

The value of the line integral does not depend on the parametrization of the curve provided that the curve is traversed exactly once as t increases from a to b.

Example 1. Evaluate the line integral $\int_C x ds$, where C is a given by $x = t^3$, $y = t$, $0 \leq t \leq 1$.

Suppose now that C is a piecewise-smooth curve; that is, C is a union of a finite number of smooth functions C_1, C_2,...,C_n. Then

$$\int_C f(x,y)ds = \int_{C_1} f(x,y)ds + \int_{C_2} f(x,y)ds + ... + \int_{C_n} f(x,y)ds$$

Example 2. Evaluate $\int_C (x+y)ds$ if C consists of line segments from (1,0) to (0,1), from (0,1) to (0,0), and from (0,0) to (0,1).

Physical interpretation of a line integral $\int_C f(x,y)ds$. Suppose that $\rho(x,y)$ represents the linear density at a point (x,y) of a thin wire shaped like a curve C. Then the **mass** of wire is

$$m = \int_C \rho(x,y)ds$$

The **center of mass** of the wire with density function ρ is at the point $(\bar{x}, \bar{y})$, where

$$\bar{x} = \frac{1}{m}\int_C x\rho(x,y)ds \qquad \bar{y} = \frac{1}{m}\int_C y\rho(x,y)ds$$

Example 3. A thin wire is bent into the shape of a semicircle $x^2 + y^2 = 4$, $x \geq 0$. If the linear density is a constant k, find the mass and center of mass of the wire.

Line integrals of f along C with respect to x and y:

$$\int_C f(x,y)dx = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta x_i, \qquad \Delta x_i = x_i - x_{i-1}$$

$$\int_C f(x,y)dy = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta y_i, \qquad \Delta y_i = y_i - y_{i-1}$$

If $x = x(t)$, $y = y(t)$, then $dx = x'(t)dt$, $dy = y'(t)dt$, and

$$\int_C f(x,y)dx = \int_a^b f(x(t), y(t))x'(t)dt$$

$$\int_C f(x,y)dy = \int_a^b f(x(t), y(t))y'(t)dt$$

In general, we will write

$$\int_C P(x,y)dx + Q(x,y)dy = \int_C P(x,y)dx + \int_C Q(x,y)dy$$

Example 3. Evaluate $\int_C x\sqrt{y}dx + 2y\sqrt{x}dy$, if C consists of the arc of the circle $x^2 + y^2 = 1$ from (1,0) to (0,1) and the line segment from (0,1) to (4,3).

A given parametrization $x = x(t)$, $y = y(t)$, $a \leq t \leq b$, determines an **orientation** of a curve C, with the positive direction corresponding to increasing value of the parameter t.

If $-C$ denotes the curve consisting of the same points as C but with the opposite orientation, then we have

$$\int_{-C} f(x,y)dx = -\int_C f(x,y)dx \qquad \int_{-C} f(x,y)dy = -\int_C f(x,y)dy$$

but

$$\int_{-C} f(x,y)ds = \int_C f(x,y)ds$$

Line integrals in space.

Suppose that C is a smooth space curve given by the parametric equations

$$x = x(t), \quad y = y(t) \quad z = z(t), \quad a \leq t \leq b$$

or by a vector equation $\vec{r}(t) = x(t)\vec{\imath} + y(t)\vec{\jmath} + z(t)\vec{k}$. We define the **linear integral of f along C with respect to arc length** as

$$\int_C f(x,y,z)ds = \int_a^b f(x(t),y(t),z(t))\sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2 + \left[\frac{dz}{dt}\right]^2}dt = \int_a^b f(\vec{r}(t))|\vec{r}'(t)|dt$$

If $f(x,y,z) = 1$, then

$$\int_c ds = \int_a^b |\vec{r}'(t)dt| = L$$

Line integral along C with respect to x, y, and z can also be defined as

$$\int_C P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz = \int_a^b [P(x,y,z)x'(t) + Q(x,y,z)y'(t) + R(x,y,z)z'(t)]dt$$

Example 4. Evaluate $\int_C x^2 z ds$ if C is given by $x = \sin(2t)$, $y = 3t$, $z = \cos(2t)$, $0 \leq t \leq \pi/4$.

Example 5. Evaluate $\int_C yzdy + xydz$ if C is given by $x = \sqrt{t}$, $y = t$, $z = t^2$, $0 \leq t \leq 1$.

Line integrals of vector fields.

Definition. Let $\vec{F}$ be continuous vector field defined on a smooth curve C given by a vector function $\vec{r}(t)$, $a \leq t \leq b$. Then the **line integral of** F **along** C is

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)dt = \int_C \vec{F} \cdot \vec{T} ds$$

where $\vec{T}$ is a unit tangent vector.

If $\vec{F} = P\vec{\imath} + Q\vec{\jmath} + R\vec{k}$, then

$$\int_C \vec{F} \cdot d\vec{r} = \int_C Pdx + Qdy + Rdx$$

Example 6. Find the work done by the force field $\vec{F}(x, y, z) = xz\vec{\imath} + xy\vec{\jmath} + yz\vec{k}$ on a particle that moves along the curve $\vec{r}(t) =< t^2, -t^3, t^4 >$, $0 \leq t \leq 1$.