Chapter 14. **Vector calculus.**
Section 14.3 **The Fundamental Theorem for line integrals.**

Theorem. Let C be a smooth curve given by the vector function $\vec{r}(t)$, $a \le t \le b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

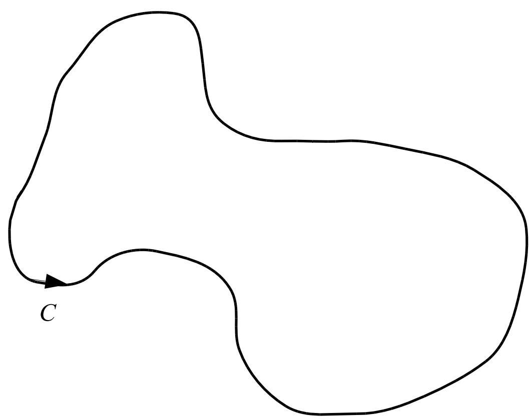
$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

Independence of path.

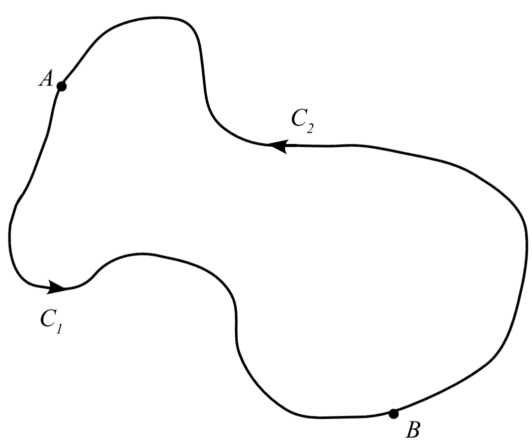
Suppose C_1 and C_2 are two piecewise-smooth curves (which are called **paths**) that have the same initial point A and the terminal point B. In general, $\int_{C_1} \vec{F} \cdot d\vec{r} \neq \int_{C_2} \vec{F} \cdot d\vec{r}$. But, according the the Theorem, if ∇f is continuous, then $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$. In other words, the line integral of a conservative vector field depends only on the initial point and terminal point of a curve.

In general, if $\vec{F}$ is a continuous vector-field with domain D, we say that the line integral is **independent of path** if $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$ for any two paths C_1 and C_2 in D that have the same initial and terminal points. **Line integrals of conservative vector fields are independent of path**.

A curve is called **closed** if its terminal point coincides with its initial point, that is $\vec{r}(a) = \vec{r}(b)$.



If $\int_C \vec{F} \cdot d\vec{r}$ is independent of path in D and C is any closed path in D, we can choose any two points A and B on C and regard C as being composed of the path C_1 from A to B followed by the path C_2 from B to A.



Then

$$\oint_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} - \int_{-C_2} \vec{F}\cdot d\vec{r} = 0$$

Also we can show that if $\oint_C \vec{F}\cdot d\vec{r} = 0$ whenever C is a closed path in D, then $\int_C \vec{F}\cdot d\vec{r}$ is independent of path in D.

Theorem. $\int_C \vec{F}\cdot d\vec{r}$ is independent of path in D if and only if $\oint_C \vec{F}\cdot d\vec{r} = 0$ for every closed path in D.

Now we assume that D is **open** (for every point P in D there is a disk with center P that lies entirely in D) and **connected** (any two points in D can be joined by a path that lies in D).

Theorem. Suppose $\vec{F}$ is a vector field that is continuous on an open connected region D. If $\int_C \vec{F}\cdot d\vec{r}$ is independent of path in D, then $\vec{F}$ is a conservative vector field on D; that is, there exists a function f such that $\nabla f = \vec{F}$.

Question: How to determine whether or not a vector field $\vec{F}$ is conservative?

Theorem. If $\vec{F}(x,y) = P(x,y)\vec{\imath} + Q(x,y)\vec{\jmath}$ is a conservative vector fields, where P and Q have continuous first-order partial derivatives on a domain D, then throughout D we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

The converse of Theorem is true only for a special type of the region.

Definition. A curve is **simple** if it does not cross itself anywhere between its endpoints.

Definition. A **simply-connected region** in the plane is a connected region D such that every simple closed curve in D encloses only points that are in D (simply-connected region contains no hole and cannot consist of two separate pieces).

Theorem. Let $\vec{F} = P\vec{\imath} + Q\vec{\jmath}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Then $\vec{F}$ is conservative.

Example 1. Determine whether or not the vector field

$$\vec{F}(x,y) = (y\cos x - \cos y)\vec{\imath} + (\sin x + x\sin y)\vec{\jmath}$$

is conservative.

Example 2.

1. If $\vec{F} =< 2xy^3, 3x^2y^2 >$, find a function f such that $\nabla f = \vec{F}$.

2. Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ along the curve C given by $\vec{r}(t) =< \sin t, t^2 + 1 >, 0 \leq t \leq \pi/2$.

Example 3.

1. If $\vec{F} =< 2xz + \sin y, x \cos y, x^2 >$, find a function f such that $\nabla f = \vec{F}$.

2. Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ along the curve C given by $\vec{r}(t) =< \cos t, \sin t, t >$, $0 \leq t \leq 2\pi$.

Example 4. Show that the line integral $\int_C 2x \sin y\, dx + (x^2 \cos y - 3y^2)\, dy$ is independent of path and evaluate the integral if C is any path from (-1,0) to (5,1).