Chapter 14. **Vector calculus.**

Section 14.6 **Parametric surfaces and their areas.**

We suppose that

$$\vec{r}(u,v) = x(u,v)\vec{\imath} + y(u,v)\vec{\jmath} + z(u,v)\vec{k}$$

is a vector-valued function defined on an region D in the uv-plane and the partial derivatives of x, y, and z with respect to u and v are all continuous. The set of all points $(x,y,z) \in \mathbb{R}^3$, such that

$$x = x(u,v), \quad y = y(u,v) \quad z = z(u,v)$$

and $(u,v) \in D$, is called a **parametric surface** S with parametric equations

$$x = x(u,v), \quad y = y(u,v) \quad z = z(u,v).$$

Example 1. Find a parametric representation for the part of the elliptic paraboloid $y = 6 - 3x^2 - 2z^2$ that lies to the right of the xz-plane.

In general, a surface given as the graph of the function $z = f(x,y)$, can always be regarded as a parametric surface with parametric equations

$$x = x, \quad y = u \quad z = f(x,y).$$

Surfaces of revolution also can be represented parametrically. Let us consider the surface S obtained by rotating the curve $y = f(x)$, $a \le x \le b$, about the x-axis, where $f(x) \ge 0$ and f' is continuous.

Let θ be the angle of rotation. If (x,y,z) is a point on S, then

$$x = x \quad y = f(x)\cos\theta \quad z = f(x)\sin\theta$$

The parameter domain is given by $a \le x \le b$, $0 \le \theta \le 2\pi$.

Example 2. Find equation for the surface generated by rotating the curve $x = 4y^2 - y^4$, $-2 \leq y \leq 2$, about the y-axis.

Tangent planes.

Problem. Find the tangent plane to a parametric surface S given by a vector function $\vec{r}(u, v)$ at a point P_0 with position vector $\vec{r}(u_0, v_0)$.

The tangent vector $\vec{r}_v$ to C_1 at P_0 is

$$\vec{r}_v = \frac{\partial x}{\partial v}(u_0, v_0)\vec{\imath} + \frac{\partial y}{\partial v}(u_0, v_0)\vec{\jmath} + \frac{\partial z}{\partial v}(u_0, v_0)\vec{k}$$

Similarly, the tangent vector $\vec{r}_u$ to C_2 at P_0 is

$$\vec{r}_u = \frac{\partial x}{\partial u}(u_0, v_0)\vec{\imath} + \frac{\partial y}{\partial u}(u_0, v_0)\vec{\jmath} + \frac{\partial z}{\partial u}(u_0, v_0)\vec{k}$$

Then the **normal vector** to the tangent plane to a parametric surface S at P_0 is the vector $\vec{r}_u \times \vec{r}_v$. If $\vec{r}_u \times \vec{r}_v \neq \vec{0}$, then S is called **smooth**.

Example 3. Find the tangent plane to the surface with parametric equations $\vec{r}(u, v) = (u + v)\vec{\imath} + u \cos v\vec{\jmath} + v \sin u\vec{k}$ at the point (1,1,0).

Surface area. Let S be a parametric surface given by a vector function $\vec{r}(u,v)$, $(u,v) \in D$. For simplicity, we start by considering a surface whose parameter domain D is a rectangle, and we partition it into subrectangles R_{ij}.

Let's choose (u_i^*, v_j^*) to be the lower left corner of R_{ij}. The part S_{ij} of the surface S that corresponds to R_{ij} has the point P_{ij} with position vector $\vec{r}(u_i^*, v_j^*)$ as one of its corners. Let

$$\vec{r}_{u_i} = \vec{r}_u(u_i^*, v_j^*), \qquad \vec{r}_{v_j} = \vec{r}_v(u_i^*, v_j^*)$$

be the tangent vectors at P_{ij}. We approximate S_{ij} by the parallelogram determined by the vectors $\Delta u_i \vec{r}_{u_i}$ and $\Delta v_j \vec{r}_{v_j}$ (this parallelogram lies in the tangent plane to S at P_{ij}). The area of this parallelogram is

$$|(\Delta u_i \vec{r}_{u_i}) \times (\Delta v_j \vec{r}_{v_j})| = |\vec{r}_{u_i} \times \vec{r}_{v_j}| \Delta u_i \Delta v_j$$

so an approximation to the area of S is

$$\sum_{i=1}^{m} \sum_{j=1}^{n} |\vec{r}_{u_i} \times \vec{r}_{v_j}| \Delta u_i \Delta v_j \to \iint_D |\vec{r}_u \times \vec{r}_v| dA \text{ as } \|P\| \to 0.$$

Definition. If a smooth parametric surface S is given by the equation $\vec{r}(u,v) = x(u,v)\vec{\imath} + y(u,v)\vec{\jmath} + z(u,v)\vec{k}$, $(u,v) \in D$ and S is covered just once at (u,v) ranges throughout the parameter domain D, then the **surface area** of S is

$$A(S) = \iint_D |\vec{r}_u \times \vec{r}_v| dA$$

where

$$\vec{r}_u = \frac{\partial x}{\partial u}\vec{\imath} + \frac{\partial y}{\partial u}\vec{\jmath} + \frac{\partial z}{\partial u}\vec{k} \qquad \vec{r}_v = \frac{\partial x}{\partial v}\vec{\imath} + \frac{\partial y}{\partial v}\vec{\jmath} + \frac{\partial z}{\partial v}\vec{k}$$

If a surface S is given by $z = f(x,y)$, $(x,y) \in D$, the parametric equations for S are

$$x = x, \qquad y = u \qquad z = f(x,y)$$

Then $\vec{r}_x = < 1, 0, f_x(x,y) >$, $\vec{r}_y = < 0, 1, f_y(x,y) >$, and

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & 0 & f_x(x,y) \\ 0 & 1 & f_y(x,y) \end{vmatrix} = -f_x\vec{\imath} - f_y\vec{\jmath} + \vec{k}$$

Then

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{1 + [f_x(x,y)]^2 + [f_y(x,y)]^2}$$

and

$$A(S) = \iint_D \sqrt{1 + [f_x(x,y)]^2 + [f_y(x,y)]^2}\, dA$$

Example 4. Find the surface area of the part of the surface $z = x + y^2$ that lies above the triangle with vertices (0,0), (1,1), and (0,1).

Example 5. Find the surface area of the part of the sphere $x^2 + y^2 + z^2 = a^2$ that lies inside the cylinder $x^2 + y^2 = ax$.