## Chapter 14. **Vector calculus.**
### Section 14.7 **Surface integrals.**

Suppose $f$ is a function of three variables whose domain include a surface $S$. We divide $S$ into patches $S_{ij}$ with area $\Delta S_{ij}$. We evaluate $f$ at a point $P_{ij}^*$ in each patch, multiply by the area $\Delta S_{ij}$, and form the sum

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij}^*)\Delta S_{ij}$$

We define the **surface integral of** $f$ **over the surface** $S$ as

$$\iint_S f(x,y,z)dS = \lim_{\|P\|\to 0}\sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij}^*)\Delta S_{ij}$$

If the surface $S$ is given by an equation $z = g(x,y)$, $(x,y)\in D$, then

$$\iint_S f(x,y,z)dS = \iint_D f(x,y,g(x,y))\sqrt{\left[\frac{\partial g}{\partial x}\right]^2 + \left[\frac{\partial g}{\partial y}\right]^2 + 1}\, dA$$

If the surface $S$ is given by vector function $\vec{r}(u,v) = x(u,v)\vec{\imath} + y(u,v)\vec{\jmath} + z(u,v)\vec{k}$, $(u,v)\in D$, then

$$\iint_S f(x,y,z)dS = \iint_D f(\vec{r}(u,v))|\vec{r}_u\times\vec{r}_v|\, dA$$

where

$$\vec{r}_u = \frac{\partial x}{\partial u}\vec{\imath} + \frac{\partial y}{\partial u}\vec{\jmath} + \frac{\partial z}{\partial u}\vec{k} \qquad \vec{r}_v = \frac{\partial x}{\partial v}\vec{\imath} + \frac{\partial y}{\partial v}\vec{\jmath} + \frac{\partial z}{\partial v}\vec{k}$$

**Example 1.**

1. Evaluate $\iint_S y\, dS$, where $S$ is the part of the plane $3x + 2y + z = 6$ that lies in the first octant.

2. Evaluate $\iint_S \sqrt{1+x^2+y^2}\,dS$, if $S$ is given by vector equation $\vec{r}(u,v) = u\cos v\vec{\imath} + u\sin v\vec{\jmath} + v\vec{k}$, $0 \le u \le 1$, $0 \le v \le \pi$.

3. Evaluate $\iint_S xy\, dS$, if $S$ is a boundary of the region enclosed by the cylinder $x^2 + z^2 = 1$ and the planes $y = 0$ and $x + y = 2$.

If a thin sheet has the shape of a surface $S$ and the density at the point $(x, y, z)$ is $\rho(x, y, z)$, then the total **mass** of the sheet is

$$m = \iint_S \rho(x, y, z) dS$$

and the **center of mass** is $(\bar{x}, \bar{y}, \bar{z})$, where

$$\bar{x} = \frac{1}{m} \iint_S x\rho(x, y, z) dS \quad \bar{y} = \frac{1}{m} \iint_S y\rho(x, y, z) dS \quad \bar{z} = \frac{1}{m} \iint_S z\rho(x, y, z) dS$$

**Oriented surfaces.**

Let us consider a surface $S$ that has a tangent plane at every point $(x, y, z)$ on $S$ (except at any boundary point).

There are two unit normal vectors $\vec{n_1}$ and $\vec{n_2} = -\vec{n_1}$ at $(x, y, z)$. If it is possible to chose a unit normal vector $\vec{n}$ at every such point $(x, y, z)$ so that $\vec{n}$ varies continuously over $S$, then $S$ is called an **oriented surface** and the given choice of $\vec{n}$ provides $S$ with an **orientation**. There are two possible orientations for any orientable surface.

For a surface $z = g(x, y)$ the orientation is given by the unit normal vector

$$\vec{n} = \frac{-\frac{\partial g}{\partial x}\vec{\imath} - \frac{\partial g}{\partial y}\vec{\jmath} + \vec{k}}{\sqrt{\left[\frac{\partial g}{\partial x}\right]^2 + \left[\frac{\partial g}{\partial y}\right]^2 + 1}}$$

Since the $\vec{k}$-component is positive, this gives the *upward* orientation of the surface.

If $S$ is a smooth orientable surace given in parametric form by a vector function $\vec{r}(u, v)$, then its orientation is given by a unit normal vector

$$\vec{n} = \frac{\vec{r_u} \times \vec{r_v}}{|\vec{r_u} \times \vec{r_v}|}$$

For a **closed surface**, the positive orientation is the one for which the normal vectors point *outward* from $S$, the inward-pointing normals give the negative orientation.

**Surface integrals of vector fields.**

**Definition.** If $\vec{F}$ is a continuous vector-field defined on an oriented surface $S$ with normal vector $\vec{n}$, then the **surface integral of $F$ over $S$** is

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\, dS$$

This integral is also called the **flux** of $\vec{F}$ across $S$.

If $\vec{F} =< P(x,y,z), Q(x,y,z), R(x,y,z) >$ and the surface $S$ is given by an equation $z = g(x,y)$, $(x,y) \in D$, then

$$\vec{n} = \frac{-\dfrac{\partial g}{\partial x}\vec{\imath} - \dfrac{\partial g}{\partial y}\vec{\jmath} + \vec{k}}{\sqrt{\left[\dfrac{\partial g}{\partial x}\right]^2 + \left[\dfrac{\partial g}{\partial y}\right]^2 + 1}}$$

and

$$\iint\limits_S \vec{F} \cdot dS = \iint\limits_S \vec{F} \cdot \vec{n}\, dS = \iint\limits_D (P\vec{\imath} + Q\vec{\jmath} + R\vec{k}) \cdot \frac{-\dfrac{\partial g}{\partial x}\vec{\imath} - \dfrac{\partial g}{\partial y}\vec{\jmath} + \vec{k}}{\sqrt{\left[\dfrac{\partial g}{\partial x}\right]^2 + \left[\dfrac{\partial g}{\partial y}\right]^2 + 1}} \sqrt{\left[\dfrac{\partial g}{\partial x}\right]^2 + \left[\dfrac{\partial g}{\partial y}\right]^2 + 1}\, dA$$

or

$$\iint\limits_S \vec{F} \cdot dS = \iint\limits_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA$$

If the surface $S$ is given by vector function $\vec{r}(u,v) = x(u,v)\vec{\imath} + y(u,v)\vec{\jmath} + z(u,v)\vec{k}$, $(u,v) \in D$, then

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

and

$$\iint\limits_S \vec{F} \cdot dS = \iint\limits_S \vec{F} \cdot \vec{n}\, dS = \iint\limits_D \vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} |\vec{r}_u \times \vec{r}_v|\, dA$$

or

$$\iint\limits_S \vec{F} \cdot d\vec{S} = \iint\limits_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\, dA$$

**Example 2.** Find the flux of the vector field $\vec{F} = x^2y\vec{\imath} - 3xy^2\vec{\jmath} + 4y^3\vec{k}$ across the surface $S$, if $S$ is the part of the elliptic paraboloid $z = x^2 + y^2 - 9$ that lies below the rectangle $0 \le x \le 2$, $0 \le y \le 1$ and has downward orientation.

**Example 3.** A fluid has density 1500 and velocity field

$$\vec{v} = -y\vec{\imath} + x\vec{\jmath} + 2z\vec{k}$$

Find the rate of flow outward through the sphere $x^2 + y^2 + z^2 = 25$.