MATH 251–507, 508, 511

Midterm Exam I, October 7, 2010

Show all your work neatly and concisely and **clearly indicate your final answer**. The use of a calculator, laptop or computer is prohibited.

1. *[10 pts.]* Find an equation of the plane that passes through the points $(-1,2,0)$, $(2,0,1)$, and $(-5,3,1)$.

   SOLUTION. Let $A(-1,2,0)$, $B(2,0,1)$, and $C(-5,3,1)$. Two nonparallel vectors that lie in the plane are

$$\vec{AB} =< 2-(-1), 0-2, 1-0 >=< 3,-2,1 >$$
$$\vec{AC} =< -5-(-1), 3-2, 1-0 >=< -4,1,1 >$$

   The vector orthogonal to the plane is

$$\vec{n} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 3 & -2 & 1 \\ -4 & 1 & 1 \end{vmatrix} = -3\vec{\imath} - 7\vec{\jmath} + 11\vec{k}$$

   Thus, the equation of the plane is

$$-3(x+1) - 7(y-2) + 11(z-0) = 0$$

   or

$$3x + 7y - 11z - 11 = 0$$

2. *[13 pts.]* Find the length of the curve $\vec{r}(t) =< 6t, 3\sqrt{2}\, t^2, 2t^3 >$, $0 \le t \le 1$.

   SOLUTION. $\vec{r}'(t) =< 6, 6\sqrt{2}t, 6t^2 >= 6 < 1, \sqrt{2}t + t^2 >$

$$|\vec{r}'(t)| = 6\sqrt{1+2t^2+t^4} = 6\sqrt{(1+t^2)^2} = 6(1+t^2)$$

   Then

$$L = \int_0^1 6(1+t^2)dt = 6\left(t + \frac{t^3}{3}\right)\Big|_0^1 = 8$$

3. *[13 pts.]* Find the curvature of the curve $\vec{r}(t) =< t^2+2, t^2-4t, 2t >$.

   SOLUTION. $\kappa(t) = \dfrac{|\vec{r'} \times \vec{r''}|}{|\vec{r'}|^3}$.

$$\vec{r'}(t) =< 2t, 2t-4, 2 >, \qquad |\vec{r'}(t)| = 2\sqrt{t^2+(t-2)^2+1} = 2\sqrt{2t^2-4t+5}$$
$$\vec{r''}(t) =< 2,2,0 >$$
$$\vec{r'} \times \vec{r''} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 2t & 2t-4 & 2 \\ 2 & 2 & 0 \end{vmatrix} = -4\vec{\imath} + 4\vec{\jmath} + 8\vec{k} = 4 < -1,1,2 >$$
$$|\vec{r'} \times \vec{r''}| = 4\sqrt{1+1+4} = 4\sqrt{6}$$

   Thus,

$$\kappa(t) = \frac{4\sqrt{6}}{8(2t^2-4t+5)^{3/2}} = \frac{\sqrt{6}}{2(2t^2-4t+5)^{3/2}}$$

4. *[12 pts.]* Let $f(x, y, z) = x + \ln(y^2 + z^2)$. Find a vector in the direction in which f increases most rapidly at the point $P(2, 1, 1)$.

SOLUTION. The function f is increasing most rapidly in the direction in ∇f

$$\nabla f =< 1, \frac{2y}{y^2 + z^2}, \frac{2z}{y^2 + z^2} >$$

$$\nabla f(2, 1, 1) =< 1, \frac{2}{2}, \frac{2}{2} >=< 1, 1, 1 >$$

5. *[12 pts.]* Find **parametric** equations of the normal line to the surface $xy^2z^3 = 12$ at the point $(3, 2, 1)$.

SOLUTION. Let $F(x, y, z) = xy^2z^3 - 12$. Then

$$\frac{\partial F}{\partial x} = y^2z^3 \qquad \frac{\partial F}{\partial x}(3, 2, 1) = 4$$

$$\frac{\partial F}{\partial y} = 2xyz^3 \qquad \frac{\partial F}{\partial y}(3, 2, 1) = 12$$

$$\frac{\partial F}{\partial z} = 3xy^2z^2 \qquad \frac{\partial F}{\partial z}(3, 2, 1) = 36$$

Thus, parametric equation of the normal line are

$$x = 3 + 4t, \quad y = 2 + 12t \quad z = 1 + 36t$$

6. *[12 pts.]* Use the differential to estimate

$$\sqrt{5(1.04)^2 + 4(0.95)^2}$$

SOLUTION. $f(a + \Delta x, b + \Delta y) \approx f(a, b) + \frac{\partial f}{\partial x}(a, b)\Delta x + \frac{\partial f}{\partial y}(a, b)\Delta y$

$$f(x, y) = \sqrt{5x^2 + 4y^2}, \quad a = 1,\ b = 1, \quad \Delta x = 0.04,\ \Delta y = -0.05$$

$$\frac{\partial f}{\partial x} = \frac{5x}{\sqrt{5x^2 + 4y^2}}, \frac{\partial f}{\partial y} = \frac{4y}{\sqrt{5x^2 + 4y^2}}, \quad \frac{\partial f}{\partial x}(1, 1) = \frac{5}{3}, \frac{\partial f}{\partial y} = \frac{4}{3}$$

Thus

$$\sqrt{5(1.04)^2 + 4(0.95)^2} \approx f(1, 1) + \frac{\partial f}{\partial x}(1, 1)\Delta x + \frac{\partial f}{\partial y}(1, 1)\Delta y = 5 + \frac{5}{3}(0.04) + \frac{4}{3}(-0.05) = 3$$

7. *[13 pts.]* Given that $w = 5x^3y + x$ where $x = t\tan s$, $y = t + \ln s$, use the Chain Rule to find $\frac{\partial w}{\partial s}$.

SOLUTION.

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} = (15x^2y + 1)\sec^2 s + 5x^3\frac{1}{s}$$

8. *[15 pts.]* Find the dimensions of the rectangular box with largest volume if the total surface area is given as 64 cm^2.

SOLUTION. Let's dimensions of the box are $l \times w \times h$. Then the surface area

$$A = 2(lw + lh + wh) = 64$$

or

$$lw + lh + wh = 32$$

So,

$$h = \frac{32 - lw}{l + w}$$

The volume of the box is

$$V = lwh = lw\frac{32 - lw}{l + w} = \frac{32lw - l^2w^2}{l + w}$$

$$\frac{\partial V}{\partial l} = \frac{(32w - 2lw^2)(l + w) - 32lw + l^2w^2}{(l + w)^2} = \frac{w^2(32 - l^2 - 2lw)}{(l + w)^2} = 0$$

$$\frac{\partial V}{\partial w} = \frac{l^2(32 - w^2 - 2lw)}{(l + w)^2} = 0$$

Since $l \neq 0$ and $w \neq 0$,

$$\begin{cases} 32 - l^2 - 2lw = 0 \\ 32 - w^2 - 2lw = 0 \end{cases}$$

If we subtract the second equation from the first equation, we will get

$$w^2 - l^2 = 0$$

or

$$l = w$$

Plugging $l = w$ into the first equation gives

$$32 - l^2 - 2l^2 = 0$$

or

$$l = \sqrt{\frac{32}{3}} = w$$

So,

$$h = \frac{32 - lw}{l + w} = \frac{32 - \frac{32}{3}}{2\sqrt{\frac{32}{3}}} = \frac{32}{3}$$

Bonus Problem (*[10 pts]*, **no partial credit**). The plane $y + z = 3$ intersects the cylinder $x^2 + y^2 = 5$ in an ellipse. Find **symmetric** equations for the tangent line to this ellipse at the point $(1, 2, 1)$.

SOLUTION. Let $F(x, y, z) = y + z - 3$, $G(x, y, z) = x^2 + y^2 - 5$,

$$\nabla F(x, y, z) =< 0, 1, 1 >, \quad \nabla F(1, 2, 1) =< 0, 1, 1 >$$

$$\nabla G(x,y,z) =< 2x, 2y, 0 >, \nabla G(1,2,1) =< 2,4,0 >$$

$$\nabla F(1,2,1) \times \nabla G(1,2,1) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 0 & 1 & 1 \\ 2 & 4 & 0 \end{vmatrix} =< -4, -2, -2 >$$

Thus, **symmetric** equations for the tangent line to this ellipse at the point $(1,2,1)$ are

$$\frac{x-1}{-4} = \frac{y-2}{-2} = \frac{z-1}{-2}$$

or

$$\frac{x-1}{2} = y - 2 = z - 1$$