HW #4 (key).

Section ~~3.4~~ 2.6.

#3. $(3x^2-2xy+2)+(6y^2-x^2+3)y'=0.$

$$(3x^2-2xy+2)dx+(6y^2-x^2+3)dy=0.$$

$$M(x,y)=3x^2-2xy+2, \quad \frac{\partial M}{\partial y}=-2x$$

exact.

$$N(x,y)=6y^2-x^2+3, \quad \frac{\partial N}{\partial x}=-2x$$

$$F(x,y): \begin{cases} \dfrac{\partial F}{\partial x}=3x^2-2xy+2 \\ \dfrac{\partial F}{\partial y}=6y^2-x^2+3 \end{cases}$$

$$F(x,y)=3y^3-x^2y+3y+g(x).$$

$$\frac{\partial F}{\partial x}=-2xy+g'(x)=3x^2-2xy+2$$

$$g'(x)=3x^2+2$$

$$g(x)=x^3+2x+C.$$

$$F(x,y)=3y^3-x^2y+3y+x^3+2x+C.$$

General solution:

$$\boxed{3y^3-x^2y+3y-x^3+2x+C=0}$$

#7. $(e^x \sin y - 2y \sin x) + (e^x \cos y + 2\cos x) y' = 0.$

$(e^x \sin y - 2y \sin x)\,dx + (e^x \cos y + 2\cos x)\,dy = 0.$

$M(x,y) = e^x \sin y - 2y \sin x, \quad \frac{\partial M}{\partial y} = e^x \cos y - 2\sin x$

$N(x,y) = e^x \cos y + 2\cos x, \quad \frac{\partial N}{\partial x} = e^x \cos y - 2\sin x$

exact.

$\frac{\partial F}{\partial x} = e^x \sin y - 2y \sin x$

$F(x,y) = e^x \sin y + 2y\cos x + g(y)$

$\frac{\partial F}{\partial y} = e^x \cos y + 2\cos x + g'(y) = e^x \cos y + 2\cos x$

$g'(y) = 0 \Rightarrow g(y) = C.$

$\boxed{e^x \sin y + 2y\cos x + C = 0}$

#11. $(x \ln y + xy) + (y \ln x + xy) y' = 0, \quad x > 0, \; y > 0.$

$\frac{\partial M}{\partial y} = \frac{x}{y} + x; \quad \frac{\partial N}{\partial x} = \frac{y}{x} + y$

$\boxed{\text{NOT exact}}$

#13. $(2x-y)+(2y-x)y'=0, y(1)=3.$

$$\underbrace{(2x-y)}_{M}dx+\underbrace{(2y-x)}_{N}dy=0.$$

$$\frac{\partial M}{\partial y}=-1, \quad \frac{\partial N}{\partial x}=-1 \qquad \text{exact.}$$

$$F(x,y): \begin{cases} \dfrac{\partial F}{\partial x}=2x-y & \Rightarrow \quad F(x,y)=x^2-xy+g(y) \\ \dfrac{\partial F}{\partial y}=2y-x & \quad \dfrac{\partial F}{\partial y}=-\cancel{y}-x+g'(y) \end{cases}$$

$$-x+g'(y)=2y-x \Rightarrow g'(y)=2y \Rightarrow$$

$$g(y)=y^2+C.$$

General solution is

$$x^2-xy+y^2+C=0.$$

$\cancel{C=4}$ plug in $x=1$, $y=3$:

$$1-3+9+C=0. \Rightarrow C=-7.$$

solve for y: $\boxed{x^2-xy+y^2-7=0}$ $\quad y=\dfrac{x+\sqrt{x^2-4(C+x^2)}}{2}$

$$y^2-xy+(x^2-7)=0.$$

$$y=\frac{x\pm\sqrt{x^2-4x^2+28}}{2}=\frac{x\pm\sqrt{28-3x^2}}{2}.$$

$$28-3x^2\geq 0$$

$$x^2\leq\frac{28}{3}, \quad \boxed{|x|\leq\sqrt{\frac{28}{3}}}$$

solution is valid on

#21.

$y + (2x - y e^y) y' = 0$, $\mu(x,y) = y$.

$y^2 dx + (2xy - y^2 e^y) dy = 0 \Rightarrow$ exact.

$\frac{\partial M}{\partial y} = 2y$, $\frac{\partial N}{\partial x} = 2y$.

$F(x,y)$: $\frac{\partial F}{\partial y} = 2xy - y^2 e^y$

$F(x,y) = xy^2 - \int y^2 e^y \, dy$

$= xy^2 - \int y^2 d(e^y)$

$= xy^2 - \left(y^2 e^y - \int 2y e^y dy \right)$

$= xy^2 - y^2 e^y + 2 \left[\int y \, d(e^y) \right]$

$= xy^2 - y^2 e^y + 2(y e^y - e^y) + g(x)$.

$F(x,y) = xy^2 - y^2 e^y + 2ye^y - e^y + g(x)$

$\frac{\partial F}{\partial x} = y^2 + g'(x) = y^2$

$g'(x) = 0$, $g(x) = C$.

$F(x,y) = xy^2 - e^y(y^2 - 2y + 2) + C$

$\cancel{\boxed{xy^2 - e^y(y-1)^2 + C = 0}}$

General solution: $\boxed{xy^2 - e^y(y^2 - 2y + 2) + C = 0}$

#26. $y' = e^{2x} + y - 1$

$dy = (e^{2x} + y - 1)dx$.

$M(x,y) = e^{2x} \cancel{+} + y - 1$

$N(x,y) = -1$.

$$\frac{M_y - N_x}{N} = \frac{1}{-1} = -1.$$

$\mu(x)$: $\frac{d\mu}{\cancel{\mu}dx} = -1 \cdot \mu$.

$$\frac{d\mu}{\mu} = -dx$$

$\ln|\mu| = -x \Rightarrow \underline{\mu = e^{-x}}$.

$[(e^{2x} + y - 1)dx - dy = 0] * e^{-x}$

$(e^{x} + ye^{-x} - e^{-x})dx - e^{-x}dy = 0$.

$\frac{\partial M}{\partial y} = e^{-x}$, $\frac{\partial N}{\partial x} = e^{-x}$. exact.

$F(x,y)$:

$\frac{\partial F}{\partial x} = e^{x} + ye^{-x} - e^{-x}$

$F(x,y) = e^{x} - ye^{-x} + e^{-x} + g(y)$

$\frac{\partial F}{\partial y} = -e^{-x} + g'(y) = -e^{-x}$

$g'(y) = 0 \Rightarrow g(y) = C$

$\boxed{e^{x} - ye^{-x} + e^{-x} + C = 0}$ or $\boxed{-e^{x} + ye^{-x} - e^{-x} + C = 0}$

Section 3.1.

#1. $y''+2y'-3y=0$

$r^2+2r-3=0$ $\quad r_1=\frac{-2+\sqrt{4+12}}{2}=\frac{-2+4}{2}=1$

$(r-1)(r+3)=0.$

$r_1=1,\ r_2=-3.$

$$\boxed{y(t)=c_1e^{t}+c_2e^{-3t}}$$

#3. $6y''-y'-y=0$

$6r^2-r-1=0.$

$r_1=\frac{1+\sqrt{1+24}}{12}=\frac{6}{12}=\frac{1}{2}.$

$r_2=\frac{1-5}{12}=-\frac{4}{12}=-\frac{1}{3}.$

$$\boxed{y(t)=c_1e^{t/2}+c_2e^{-t/3}}$$

#5. $y''+5y'=0.$

$r^2+5r=0$

$r(r+5)=0 \Rightarrow r_1=0,\ r_2=-5.$

$$\boxed{y(t)=c_1+c_2e^{-5t}}$$

#7. $y''-9y'+9y=0.$

$r^2-9r+9=0.$

$r_1=\frac{9+\sqrt{81-36}}{2}=\frac{9+\sqrt{45}}{2}=\frac{9+3\sqrt{5}}{2}$

$r_2=\frac{9-3\sqrt{5}}{2}$

$$\boxed{y(t)=c_1e^{\frac{9+3\sqrt{5}}{2}t}+c_2e^{\frac{9-3\sqrt{5}}{2}t}}$$

#9.

$$y'' + y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 1$$

$$r^2 + r - 2 = 0.$$

$$(r+2)(r-1) = 0. \qquad r_1 = -2, \ r_2 = 1.$$

$$y(t) = c_1 e^{-2t} + c_2 e^{t}, \quad y'(t) = -2c_1 e^{-2t} + c_2 e^{t}$$

$$\begin{aligned} y(0) &= \\ y'(0) &= \end{aligned} \begin{cases} c_1 + c_2 = 1. \\ -2c_1 + c_2 = 1 \end{cases}$$

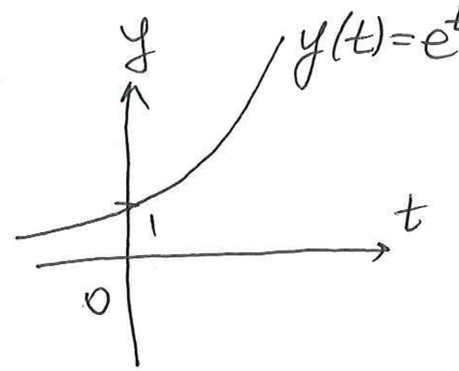
$$c_2 = 1 - c_1$$

$$-2c_1 + 1 - c_1 = 1$$

$$c_1 = 0, \ c_2 = 1.$$

$$y(t) = e^{t}$$

$$\lim_{t\to\infty} y(t) = \infty.$$

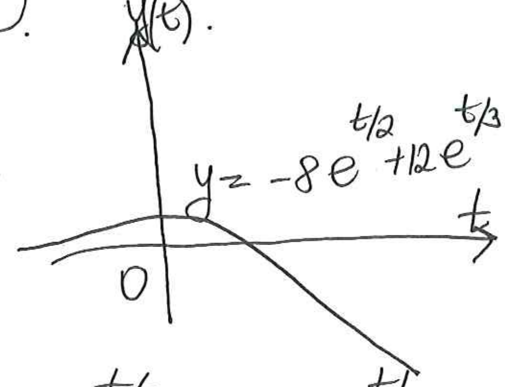


#11. $6y'' - 5y' + y = 0, \ y(0) = 4, \ y'(0) = 0.$

$$6r^2 - 5r + 1 = 0.$$

$$r_1 = \frac{5 + \sqrt{25 - 24}}{12} = \frac{6}{12} = \frac{1}{2}.$$

$$r_2 = \frac{5 - 1}{12} = \frac{4}{12} = \frac{1}{3}$$



$$y(t) = c_1 e^{t/2} + c_2 e^{t/3}, \quad y'(t) = \frac{c_1}{2} e^{t/2} + \frac{c_2}{3} e^{t/3}.$$

$$\begin{aligned} y(0) &= c_1 + c_2 = 4, \\ y'(0) &= \frac{c_1}{2} + \frac{c_2}{3} = 0. \end{aligned} \Rightarrow \begin{cases} c_1 + c_2 = 4 \\ 3c_1 + 2c_2 = 0. \end{cases} \qquad c_1 = 4 - c_2.$$

$$3(4 - c_2) + 2c_2 = 0.$$

$$12 - 3c_2 + 2c_2 = 0, \quad c_2 = 12, \ c_1 = -8$$

$$\boxed{y(t) = -8e^{t/2} + 12e^{t/3}} \qquad \boxed{\lim_{t\to\infty} y(t) = -\infty}$$

#13. $y''+5y'+3y=0, \quad y(0)=1, \; y'(0)=0$

$$r^2+5r+3=0$$

$$r_1 = \frac{-5+\sqrt{25-12}}{2} = \frac{\sqrt{13}-5}{2}$$

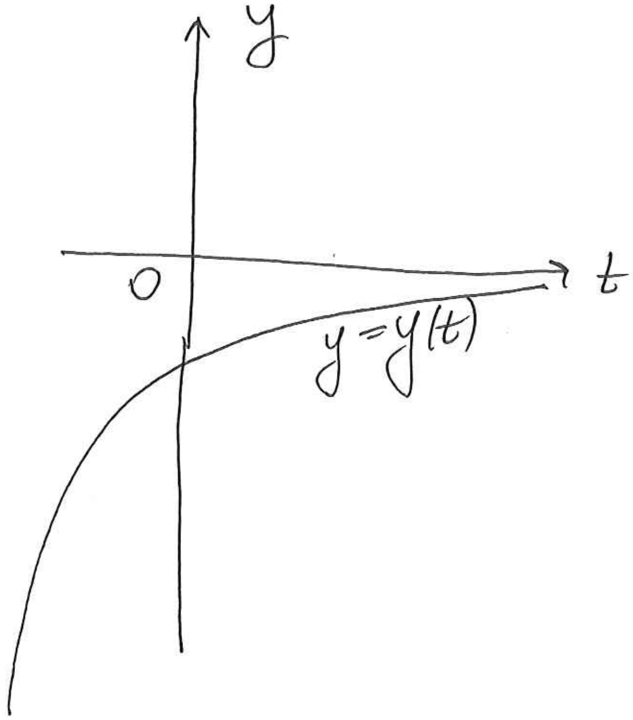
$$r_2 = -\frac{5+\sqrt{13}}{2}$$

$$y(t) = c_1 e^{\frac{\sqrt{13}-5}{2}t} + c_2 e^{-\frac{5+\sqrt{13}}{2}t}$$

$$y'(t) = c_1 \frac{\sqrt{13}-5}{2} e^{\frac{\sqrt{13}-5}{2}t} - \frac{5+\sqrt{13}}{2} c_2 e^{-\frac{5+\sqrt{13}}{2}t}$$

$$y(0) = c_1 + c_2 = 1 \Rightarrow c_2 = 1 - c_1$$

$$y'(0) = c_1 \frac{\sqrt{13}-5}{2} - \frac{\sqrt{13}+5}{2} c_2 = 0.$$



$$c_1(\sqrt{13}-5) - (\sqrt{13}+5)(1-c_1) = 0$$

$$c_1(\sqrt{13}-5+\sqrt{13}+5) = \sqrt{13}+5.$$

$$c_1 = \frac{5+\sqrt{13}}{2\sqrt{13}}$$

$$c_2 = 1 - \frac{5+\sqrt{13}}{2\sqrt{13}} = \frac{\sqrt{13}-5}{2\sqrt{13}}.$$

$$y(t) = \frac{5+\sqrt{13}}{2\sqrt{13}} e^{\frac{\sqrt{13}-5}{2}t} + \frac{\sqrt{13}-5}{2\sqrt{13}} e^{-\frac{5+\sqrt{13}}{2}t}$$

$$\boxed{\lim_{t\to\infty} y(t) = 0.}$$

#15. $y''+8y'-9y=0, \quad y(1)=1, \quad y'(1)=0.$

$r^2+8r-9=0, \quad (r-1)(r+9)=0.$

$r_1=1, \; r_2=-9$

$y(t)=c_1e^{t}+c_2e^{-9t}$

$$\begin{cases} y(1)= c_1e+c_2e^{-9}=1 \\ y'(1)= c_1e-9c_2e^{-9}=0. \end{cases}$$

$$10c_2e^{-9}=1$$

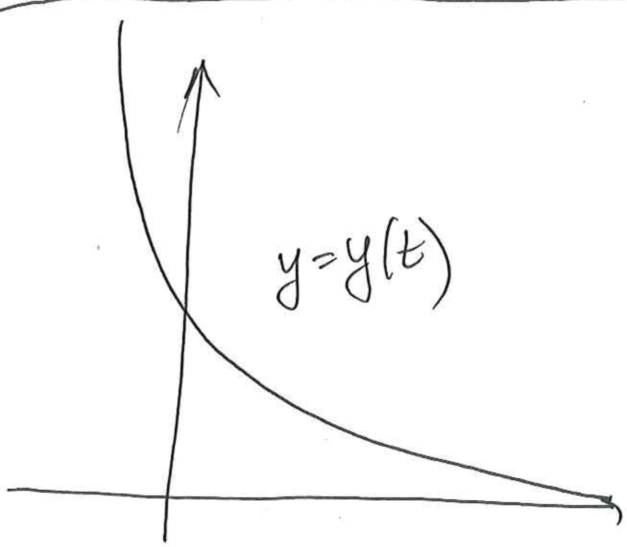
$$c_2=\frac{1}{10e^{-9}}=\frac{e^9}{10}.$$

$$c_1e=1-c_2e^{-9}$$

$$=1-\cancel{\frac{c_2}{c_1}}\frac{1}{10}=\frac{9}{10}.$$

$$c_1=\frac{9}{10}e^{-1}$$

$$\boxed{y(t)=\frac{9}{10}e^{t}e^{-1}+\frac{e^9}{10}e^{-9t}}$$



$\lim_{t\to\infty} y(t)=0.$

section 3.2.

#1-6.

#1. $W(e^{2t}, e^{-3t/2}) = \begin{vmatrix} e^{2t} & e^{-3t/2} \\ 2e^{2t} & -\frac{3}{2}e^{-3t/2} \end{vmatrix} = -\frac{7}{2}e^{t/2}$

#2. $W(\cos t, \sin t) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = 1$

#3. $W(e^{-2t}, te^{-2t}) = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2te^{-2t} \end{vmatrix}$

$= e^{-4t}$

#4. $W(x, xe^{x}) = \begin{vmatrix} x & xe^{x} \\ 1 & e^{x} + xe^{x} \end{vmatrix} = x^2 e^{x}$

#5. $W(e^{t}\sin t, e^{t}\cos t) = \begin{vmatrix} e^{t}\sin t & e^{t}\cos t \\ e^{t}\sin t + e^{t}\cos t & e^{t}\cos t - e^{t}\sin t \end{vmatrix}$

$= -e^{2t}$

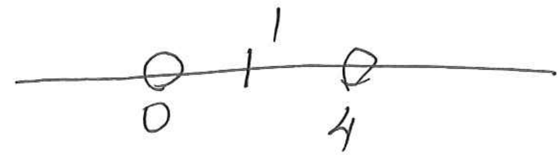
#6. $W(\cos^2\theta, 1+\cos 2\theta) = 0$, because

$\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$.

#9. $t(t-4)y'' + 3ty' + 4y = 2, \quad y(3)=0, \quad y'(3) = -1.$

$$y'' + \frac{3t}{t(t-4)}\, y' + \frac{4y}{t(t-4)} = \frac{2}{t(t-4)}$$

continuous if $t \neq 0$ & $t \neq 4$.

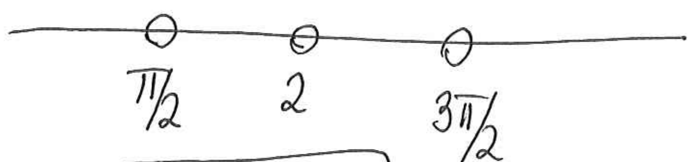


$$\boxed{(0, 4)}$$

#12. $(x-2)y'' + y' + (x-2)(\tan x)y = 0, \quad y(3)=1, \; y'(3) = 2.$

$$y'' + \frac{y'}{x-2} + (\tan x)y = 0.$$

continuous if $x \neq 2$ and $x \neq \frac{\pi}{2} + \pi n, n \in \mathbb{Z}$.



$$\boxed{\left(2, \frac{3\pi}{2}\right)}$$

#17. $W(f,g) = 3e^{4t}, \quad f(t) = e^{2t}, \quad g(t) = ?$

$$W(f,g) = \begin{vmatrix} e^{2t} & g(t) \\ 2e^{2t} & g'(t) \end{vmatrix} = 3e^{4t}$$

$e^{2t} g'(t) - 2e^{2t} g(t) = 3e^{4t}$

$g'(t) - 2g(t) = 3e^{2t}$ — linear

integrating factor

$\frac{dI}{dt} = -2I \Rightarrow \frac{dI}{I} = -2dt \Rightarrow I = e^{-2t}$

$e^{-2t} \cdot g(t) = \int 3\cancel{e^{4t}} dt = \frac{3}{4} t \cancel{e^{4t}} + C$

$\boxed{g(t) = \frac{3t}{4} e^{2t} + Ce^{+2t}}$

#28. $y'' - y' - 2y = 0.$

(a) $W(\cancel{y} e^{-t}, e^{2t}) \neq 0$

plug in $y = e^{-t}$, $y = e^{2t}$.

$\boxed{\text{YES}}$

(b) $\boxed{\text{YES.}}$.

(c) $[y_1, y_3], [y_1, y_4]$ YES.

$[y_2, y_3], [y_4, y_5]$ NO.

#35. $t^2 y'' - 2y' + (3+t)y = 0$., $W(y_1, y_2)(2) = 3$.

$W(y_1, y_2)(4) = ?$

$W(y_1, y_2) = Ce^{+\int \frac{2}{t^2} dt} = Ce^{-\frac{2}{t}}$

$W(y_1, y_2)(2) = Ce^{-1} = 3$, $C = 3e^{1/2} = \cancel{3e}\ 3e$

$W(y_1, y_2) = 3e\, e^{-2/t}$; $W(4) = 3e\, e^{-1/2} = \boxed{3e^{1/2}}$

Section 3.3

R-la.

#1. $e^{1+2i} = e(\cos 2 + i\sin 2)$

#2 $e^{2-3i} = e^{2}(\cos 3 - i\sin 3)$

#3 $e^{i\pi} = \cos\pi + i\sin\pi = -1$

#4 $e^{2-\frac{\pi}{2}i} = e^{2}(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}) = -ie^{2}$

#5 $2^{1-i} = e^{(1-i)\ln 2} = e^{\ln 2}(\cos(\ln 2) - i\sin(\ln 2))$

#6 $\pi^{-1+2i} = e^{(-1+2i)\ln\pi}$

$= e^{-\ln\pi}[\cos(2\ln\pi) + i\sin(2\ln\pi)]$.

#7. $y'' - 2y' + 2y = 0$

$r^2 - 2r + 2 = 0$

$r_1 = \frac{2+\sqrt{4-8}}{2} = \frac{2+\sqrt{-4}}{2} = 1 + \cancel{4}2i$

$r_2 = \overline{r_1} = 1 - 2i$.

$$\boxed{y(t) = e^{t}(\cos 2t + i\sin 2t)}$$

#9. $y'' + 2y' - 8y = 0$.

$r^2 + 2r - 8 = 0$.

$r_1 = \frac{-2+\sqrt{4+32}}{2} = \frac{-2+6}{2} = \cancel{3}2$

$r_2 = \frac{-2-6}{2} = -4$.

$(r+4)(r-2) = 0$.

$$\boxed{y(t) = c_1 e^{-4t} + c_2 e^{2t}}$$

#11. $y'' + 6y' + 13y = 0.$

$r^2 + 6r + 13 = 0.$

$$r_1 = \frac{-6 + \sqrt{36 - 52}}{2} = \frac{-6 + i\sqrt{16}}{2} = -3 + 2i.$$

$$\boxed{y(t) = e^{-3t}\left(c_1 \cos 2t + c_2 \sin 2t\right)}$$

#13. $y'' + 2y' + 1.25y = 0.$

$1.25 = 1\frac{25}{100} = 1\frac{1}{4} = \frac{5}{4}$

$r^2 + 2r + \frac{5}{4} = 0.$

$$r_1 = \frac{-2 + \sqrt{4-5}}{2} = \frac{-2+i}{2} = -1 + \frac{1}{2}i$$

$$\boxed{y(t) = e^{-t}\left[c_1 \cos \frac{t}{2} + c_2 \sin \frac{t}{2}\right]}$$

#15. $y'' + y' + \frac{5}{4}y = 0.$

$r^2 + r + \frac{5}{4} = 0.$

$$r_1 = \frac{-1 + \sqrt{1-5}}{2} = \frac{-1 + \sqrt{-4}}{2} = -\frac{1}{2} + i$$

$$\boxed{y(t) = e^{-\frac{t}{2}}\left[c_1 \cos t + c_2 \sin t\right]}$$

#17. $y''+4y=0, \quad y(0)=0, \quad y'(0)=1$

$r^2+4=0, \quad r=\pm 2i.$

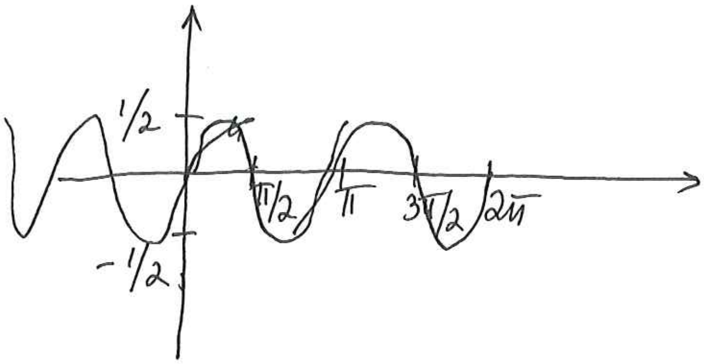
$y(t)=c_1\cos 2t + c_2 \sin 2t.$

$y'(t)=-2c_1 \sin 2t + 2c_2 \cos 2t.$

$y(0)=c_1=0$

$y'(0)=2c_2=1, \quad c_2=\frac{1}{2}.$

$y(t)=\frac{1}{2}\sin 2t$



$y(t)$ oscillates between $-\frac{1}{2}$ and $\frac{1}{2}$.

#19. $y''-2y'+5y=0, \quad y\left(\frac{\pi}{2}\right)=0, \quad y'\left(\frac{\pi}{2}\right)=2.$

$r^2-2r+5=0.$

$r_1=\frac{2+\sqrt{4-20}}{2}=\frac{2+\sqrt{-16}}{2}=\frac{2+4i}{2}=1+2i.$

$y(t)=e^t(c_1\cos 2t + c_2 \sin 2t)$

$y'(t)=e^t(c_1\cos 2t + c_2\sin 2t)+e^t(-2c_1\sin 2t + 2c_2\cos 2t)$

$y\left(\frac{\pi}{2}\right)=e^{\pi/2}(-c_1)=0 \Rightarrow c_1=0.$

$y'(\pi/2)=e^{\pi/2}(-c_1)+e^{\frac{\pi}{2}}(-2c_2)=2.$

$c_2=-e^{-\pi/2}.$

$y(t)=-e^{t-\pi/2}\sin 2t$

growing oscillation.

#21. $y'' + y' + \frac{5}{4}y = 0, \quad y(0) = 3, \quad y'(0) = 1.$

$$r^2 + r + \frac{5}{4} = 0.$$

$$r_1 = \frac{-1 + \sqrt{1-5}}{2} = \frac{-1+2i}{2} = -\frac{1}{2} + i.$$

$$y(t) = e^{-t/2}(c_1 \cos t + c_2 \sin t).$$

$$y(0) = \boxed{c_1 = 3.}$$

$$y'(t) = -\frac{1}{2}e^{-t/2}(c_1 \cos t + c_2 \sin t) + e^{-t/2}(-c_1 \sin t + c_2 \cos t)$$

$$y'(0) = -\frac{1}{2}c_1 + c_2 = 1$$

$$c_2 = 1 + \frac{c_1}{2} = 1 + \frac{3}{2} = \frac{5}{2}.$$

$$y(t) = \frac{1}{2}e^{-\frac{t}{2}}(3\cos t + 5\sin t)$$

decaying oscillation.