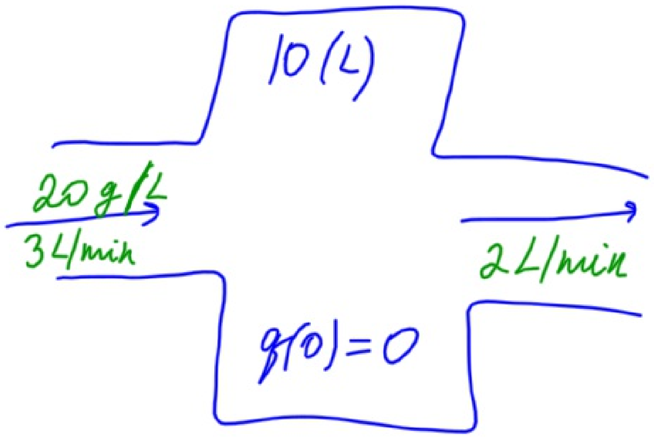
1. A large tank initially contains 10 L of fresh water. A brine containing 20 g/L of salt flows into the tank at a rate of 3 L/min. The solution inside the tank is kept well stirred and flows out of the tank at ~~the~~ a rate of 2 L/min. Determine the concentration of salt in the tank as a function of time.



q(t) is the mass of salt in the tank at time t.

$$\frac{dq}{dt} = \text{rate in} - \text{rate out}$$

$$\text{rate in} = 3(20)$$

$$\text{rate out} = \frac{q(t)}{10+(3-2)t} \cdot 2$$

$$\frac{dq}{dt} = 60 - \frac{2q(t)}{10+t} \qquad \text{IVP}$$

$$q(0)=0$$

Solve for q(t).

So $\frac{q(t)}{10+t}$ concentration

3. An object with temperature 150^0 is placed in a freezer whose temperature is 30^0. Assume that the temperature of the freezer remains essentially constant.

(a) If the object is cooled to 120^0 after 8 min, what will its temperature be after 18 min?

$T(t)$ – temperature at time t

$\frac{dT}{dt} = k(30-T)$ $\quad T(0)=150$

$T(8)=120$

Find $T(18)=?$

$\frac{dT}{dt} = -k(T-30)$

$\frac{dT}{T-30} = -k\,dt$

$\ln|T-30| = -kt+C$

$T-30 = Ce^{-kt}$

$T = 30+Ce^{-kt}$

$30+C=150$

$C=120$

$T(t)=30+120e^{-kt}$

Solve for k

$T(8) = 30+120e^{-8k}=120$

$e^{-8k}=\frac{3}{4}$

$k=-\frac{1}{8}\ln\left(\frac{3}{4}\right) \approx 0.036$

$T(t)=30+120e^{-0.036t}$

$T(18)=\ldots$

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(b) When will its temperature be 60^0?

find t such that $T(t)=60$

4. Determine (without solving the problem) an interval in which the solution to the initial value problem

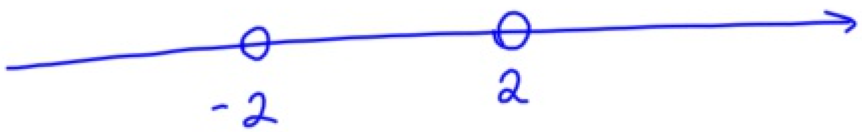
$$(4 - t^2)y' + 2ty = 3t^2, \quad y(1) = -3$$

is certain to exist.

$\boxed{(-2, 2)}$

$$y' + \frac{2t}{4-t^2} y = \frac{3t^2}{4-t^2}$$

continuous if $t \neq \pm 2$



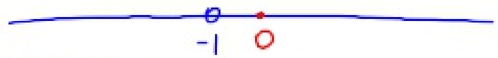
5. Solve the initial value problem

$$y' = \frac{t^2}{1+t^3}, \quad y(0) = y_0$$

and determine how the interval in which the solution exists depends on the initial value y_0.

$$f(t,y) = \frac{t^2}{1+t^3} \quad \text{continuous if } t \neq -1$$

$$\frac{\partial f}{\partial y} = 0$$



$$y' = \frac{t^2}{1+t^3}, \quad y(t_0) = y_0$$

if $t_0 > -1$, $(-1, \infty)$

if $t_0 < -1$, $(-\infty, -1)$

$0 > -1$, so $(-1, \infty)$ does not depend on y_0

Solve the initial value problem:

$$\frac{dy}{dt} = \frac{t^2}{1+t^3}$$

$$dy = \frac{t^2}{1+t^3}\, dt$$

$$y(t) = \frac{1}{3} \ln |1+t^3| + C$$

$$y(0) = C = y_0$$

$y(t) = \frac{1}{3} \ln |1+t^3| + y_0$ exists everywhere except $t = -1$.

6. Solve the following initial value problem

$$\sqrt{y}dt + (1+t)dy = 0 \quad y(0) = 1.$$

7. Find the general solution to the equation

$$(t^2 - 1)y' + 2ty + 3 = 0$$

$$y' + \frac{2t}{t^2-1} y = -\frac{3}{t^2-1}$$

$$p(t) = \frac{2t}{t^2-1} \; ; \quad g(t) = -\frac{3}{t^2-1}$$

Integrating factor:

$$\frac{d\mu(t)}{dt} = +\frac{2t\mu}{t^2-1}$$

$$\frac{d\mu}{\mu} = \frac{2t}{t^2-1} dt$$

$$\mu(t) = t^2-1$$

$$(t^2-1) y(t) = \int -\frac{3}{t^2-1} (t^2-1) dt$$

$$= -3t + C$$

$$\boxed{y(t) = -\frac{3t}{t^2-1} + \frac{C}{t^2-1}}$$

Given the differential equation

$$\frac{dy}{dt} = 7y - y^2 - 10$$

(a) Find the equilibrium solutions.

(b) Sketch the phase line and determine whether the equilibrium solutions are stable, unstable, or semistable.

3a) $7y - y^2 - 10 = 0$

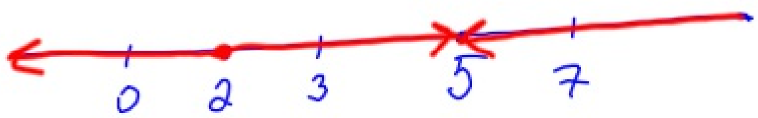
$y^2 - 7y + 10 = 0$

$(y-5)(y-2) = 0$

$y_1 = 2$ unstable

$y_2 = 5$ stable

3b)



$f(0) = -10 < 0$

$f(3) = 21 - 9 - 10 > 0$

$f(7) = 49 - 49 - 10 < 0$

$$(uvw)' = u'vw + uv'w + uvw'$$

8. Solve the initial value problem

$$\underbrace{(ye^{xy}\cos(2x) - 2e^{xy}\sin(2x) + 2x)}_{M(x,y)}dx + \underbrace{(xe^{xy}\cos(2x) - 3)}_{N(x,y)}dy = 0, \quad y(0) = -1$$

$$M_y = e^{xy}\cos(2x) + xye^{xy}\cos(2x) - 2xe^{xy}\sin(2x)$$

$$\parallel$$

$$N_x = e^{xy}\cos(2x) + xye^{xy}\cos(2x) - 2\sin(2x)xe^{xy}$$

Exact

$$\begin{cases} \dfrac{\partial F}{\partial x} = ye^{xy}\cos(2x) - 2e^{xy}\sin(2x) + 2x \\ \displaystyle\int \frac{\partial F}{\partial y}dy = \int (xe^{xy}\cos(2x) - 3)\,dy \end{cases}$$

$$\boxed{\int e^{xy}dy = \frac{1}{x}e^{xy} + C}$$

$$F(x,y) = x\frac{1}{x}e^{xy}\cos(2x) - 3y + h(x)$$

$$= e^{xy}\cos(2x) - 3y + h(x)$$

$$\frac{\partial F}{\partial x} = ye^{xy}\cos(2x) - 2e^{xy}\sin(2x) + h'(x)$$

$$h'(x) = 2x \quad \text{or} \quad h(x) = x^2 + C$$

$$F(x,y) = e^{xy}\cos(2x) - 3y + x^2 + C$$

General solution: $\boxed{e^{xy}\cos(2x) - 3y + x^2 + C = 0}$

$y(0) = -1$. Plug $y = -1$ and $x = 0$ into the general solution:

$$e^{(0)}\cos(0) - 3(-1) + 0 + C = 0$$

$$C = -4$$

$$\boxed{e^{xy}\cos(2x) - 3y + x^2 - 4 = 0}$$

9. Find an integrating factor for the equation

$$(3xy+y^2)\,dx + (x^2+xy)\,dy = 0$$

$$\underbrace{(3xy + y^2)}_{M(x,y)} + \underbrace{(x^2 + xy)}_{N(x,y)}y' = 0$$

and then solve the equation.

$$\frac{M_y - N_x}{N(x,y)} = \frac{3x+2y-(2x+y)}{x^2+xy}$$

$$= \frac{x+y}{x(x+y)}$$

$$= \frac{1}{x} \quad (\text{depends on } x \text{ only})$$

Integrating factor:

$$\frac{d\mu}{dx} = \frac{\mu}{x}$$

$$\frac{d\mu}{\mu} = \frac{dx}{x}$$

$$\mu(x) = x$$

$$\underbrace{(3x^2y+xy^2)}_{M(x,y)}dx + \underbrace{(x^3+x^2y)}_{N(x,y)}dy = 0$$

$$M_y = 3x^2+2xy$$
$$N_x = 3x^2+2xy$$

Exact

$$\begin{cases} \dfrac{\partial F}{\partial x} = 3x^2y+xy^2 \\ \displaystyle\int \dfrac{\partial F}{\partial y}\,dy = \int x^3+x^2y\,dy \end{cases}$$

$$F(x,y) = x^3y + \frac{x^2y^2}{2} + h(x)$$

$$\frac{\partial F}{\partial x} = 3x^2y + xy^2 + h'(x)$$

$$h'(x) = 0 \quad \text{or} \quad h(x) = C$$

$$F(x,y) = x^3y + \frac{x^2y^2}{2} + C$$

General solution:

$$\boxed{x^3y + \frac{x^2y^2}{2} + C = 0} \quad \boxed{y=0}$$

10. Solve the initial value problem

$$6y'' - 5y' + y = 0, \qquad y(0) = 4, y'(0) = 0$$

11. Find the general solution to the equation

$$4y'' - 12y' + 9y = 0$$

$4r^2 - 12r + 9 = 0$

$(2r-3)^2 = 0$

$r = \frac{3}{2}$ repeated root

General solution:

$$\boxed{y(t) = (c_1 + c_2 t)e^{\frac{3}{2}t}}$$

1. The Existence and Uniqueness Theorem guarantees that the solution to

$$x^3y'' + \frac{x}{\sin x}y' - \frac{2}{x-5}y = 0, \quad y(2) = 6, \quad y'(2) = 7$$

uniquely exists on

(a) $(-\pi, \pi)$
(b) $(0, \pi)$
(c) $(5, \infty)$
(d) $(0, 5)$

$$y'' + p(x)y' + q(x)y = g(x)$$
$$y(x_0) = y_0, \quad y'(x_0) = y_1$$

The initial value problem has a unique solution on the interval I such that:
- p, q, and g are continuous on I,
- x_0 is in I

$$\frac{x^3y'' + \frac{x}{\sin x}y' - \frac{2}{x-5}y = 0}{x^3}$$

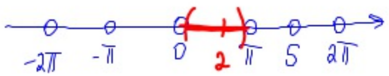
$$y'' + \frac{1}{x^2\sin x}y' - \frac{2}{x^2(x-5)}y = 0$$

$$p(x) = \frac{1}{x^2\sin x}$$

p is continuous if $x \neq 0$, $x \neq \pi n$, $n = 0, \pm1, \pm2, \ldots$ $(\sin x \neq 0)$

$$q(x) = -\frac{2}{x^2(x-5)}$$

q is continuous if $x \neq 0$, $x \neq 5$



10. The Wronskian of two functions $y_1(x) = x + 2x^2$ an $y_2(x) = 2^x$ is

(a) $2^x(1 + 4x - x(1 + 2x))$

(b) $-2^x(1 + 4x - x\ln 2(1 + 2x))$

(c) $(1 + 4x - x(1 + 2x))$

(d) $2^x(1 + 2x - x\ln 2(1 + 4x))$

$$W[y_1, y_2] = \begin{vmatrix} x+2x^2 & 2^x \\ 1+4x & 2^x \ln 2 \end{vmatrix}$$

$$= 2^x \ln 2 (x+2x^2) - 2^x(1+4x)$$

$$= 2^x(\ln 2(x+2x^2) - 1 - 4x)$$

14. Find the general solution of the equation/solve the initial value problem

(a) $y'' + 6y' + 9y = t\cos(2t)$

12. Find a general solution to the equation

$$4y'' + y' = 4x^3 + 48x^2 + 1$$

undetermined coefficients.

homogeneous eqn.

$4y'' + y' = 0$

$4r^2 + r = 0$

$r(4r+1) = 0$

$r_1 = 0, \quad r_2 = -1/4$

$y_h(x) = C_1 + C_2 e^{-\frac{1}{4}x}$

r=0 is a root of the auxiliary eqn

$y_p(x) = x(Ax^3 + Bx^2 + Cx + D)$

$= Ax^4 + Bx^3 + Cx^2 + Dx$

$y_p' = 4Ax^3 + 3Bx^2 + 2Cx + D$

$y_p'' = 12Ax^2 + 6Bx + 2C$

$4(12Ax^2 + 6Bx + 2C) + 4Ax^3 + 3Bx^2 + 2Cx + D = 4x^3 + 48x^2 + 1$

$x^3: \ 4A = 4 \qquad \boxed{A=1}$

$x^2: \ 48A + 3B = 48 \qquad 3B = 48 - 48A = 0 \qquad \boxed{B=0}$

$x: \ 24B + 2C = 0 \qquad \boxed{C=0}$

$1: \ 8C + D = 1 \qquad \boxed{D=1}$

$y_p(x) = x^4 + x$

$$\boxed{y(t) = C_1 + C_2 e^{-\frac{1}{4}x} + x^4 + x}$$

(c) $y'' + 2y' + y = 4e^{-t}$, $y(0) = 2$, $y'(0) = 1$