#19 b).

$$\vec{x}' = \begin{pmatrix} -3 & -1 \\ 1 & -1 \end{pmatrix} \vec{x}. \quad A = \begin{pmatrix} -3 & -1 \\ 1 & -1 \end{pmatrix}, \ tr A = -4, \det A = 4.$$

Characteristic equation: $\lambda^2 + 4\lambda + 4 = 0 = (\lambda+2)^2$

$\lambda = -2$ – repeated root.

Eigenvector: $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$$\begin{pmatrix} -3+2 & -1 \\ 1 & -1+2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \quad \begin{matrix} v_1 + v_2 = 0 \\ v_1 = -v_2 \end{matrix}$$

$$\vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Generalized eigenvector $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$

$$\begin{pmatrix} -3+2 & -1 \\ 1 & -1+2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \quad \begin{matrix} w_1 + w_2 = -1 \\ w_1 = -1 - w_2. \end{matrix}$$

$$\vec{w} = \begin{pmatrix} -1 - w_2 \\ w_2 \end{pmatrix} \overset{w_2 = 0}{=\!=} \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

General solution:

$$\vec{x}(t) = \cancel{c_1}\, e^{-2t} \left[c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \left(t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right) \right]$$

(0,0) is an <u>improper node, stable</u>

20.

a) $\vec{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \vec{x} + \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix}$

<u>Variation of parameters</u>

$A = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}$, $\operatorname{tr} A = -1$, $\det A = -6$

characteristic equation: $\lambda^2 + \lambda - 6 = 0 = (\lambda+3)(\lambda-2)$

$\lambda_1 = -3$, $\lambda_2 = 2$.

Corresponding eigenvectors: $\vec{v}_1 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

General solution of the corresponding homogeneous system:

$$\vec{x}_{hom}(t) = c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$$

Fundamental matrix:

$$\Psi(t) = \begin{pmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{pmatrix}, \quad \det \Psi(t) = e^{-3t}e^{2t} + 4e^{-3t}e^{2t} = 5e^{-t}.$$

$$\Psi^{-1}(t) = \frac{1}{5e^{-t}} \begin{pmatrix} e^{2t} & -e^{2t} \\ 4e^{-3t} & e^{-3t} \end{pmatrix} = \begin{pmatrix} \frac{1}{5}e^{3t} & -\frac{1}{5}e^{3t} \\ \frac{4}{5}e^{-2t} & \frac{1}{5}e^{-2t} \end{pmatrix}$$

$$\Psi^{-1}(t)\vec{g}(t) = \begin{pmatrix} \frac{1}{5}e^{3t} & -\frac{1}{5}e^{3t} \\ \frac{4}{5}e^{-2t} & \frac{1}{5}e^{-2t} \end{pmatrix} \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix} = \begin{pmatrix} \frac{1}{5}e^{t} + \frac{2}{5}e^{4t} \\ \frac{4}{5}e^{-4t} - \frac{2}{5}e^{-t} \end{pmatrix}$$

$$\int \Psi^{-1}(t)\vec{g}(t)\,dt = \begin{pmatrix} \frac{1}{5}e^{t} + \frac{1}{10}e^{4t} + \cancel{c_1}^{\,0} \\ -\frac{1}{5}e^{-4t} + \frac{2}{5}e^{-t} + \cancel{c_2}^{\,0} \end{pmatrix}$$

$$\Psi(t)\int \Psi^{-1}(t)\vec{g}(t)\,dt = \begin{pmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{pmatrix} \begin{pmatrix} \frac{1}{5}e^{t} + \frac{1}{10}e^{4t} \\ -\frac{1}{5}e^{-4t} + \frac{2}{5}e^{-t} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{5}e^{-2t} + \frac{1}{10}e^{t} - \frac{1}{5}e^{-2t} + \frac{2}{5}e^{t} \\ -\frac{4}{5}e^{-2t} - \frac{4}{10}e^{t} - \frac{1}{5}e^{-2t} + \frac{2}{5}e^{t} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}e^{t} \\ -e^{-2t} \end{pmatrix} = \vec{x}_p(t)$$

General solution: $\vec{x}(t) = \vec{x}_{hom}(t) + \vec{x}_p(t)$

$$= \boxed{c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} \frac{1}{2} e^{t} \\ -e^{-2t} \end{pmatrix}}$$

Laplace Transform. $\vec{x}(0) = \vec{0}$.

$$\mathcal{L}\{\vec{x}(t)\} = \vec{X}(s), \quad \mathcal{L}\{\vec{x}'(t)\} = s\vec{X}(s) - \vec{x}(0) = s\vec{X}(s)$$

$$s\vec{X}(s) = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \vec{X}(s) + \begin{pmatrix} \frac{1}{s+2} \\ -\frac{2}{s-1} \end{pmatrix}$$

$$\vec{x}(s) = \begin{pmatrix} s-1 & -1 \\ -4 & s+2 \end{pmatrix}^{-1} \begin{pmatrix} \frac{1}{s+2} \\ -\frac{2}{s-1} \end{pmatrix}$$

$$= \frac{1}{s^2-s+6} \begin{pmatrix} s+2 & 1 \\ 4 & s-1 \end{pmatrix} \begin{pmatrix} \frac{1}{s+2} \\ -\frac{2}{s-1} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{s+2}{(s+2)(s^2-s+6)} + \frac{2}{(s-1)(s^2-s+6)} \\ \frac{4}{(s^2-s+6)(s+2)} - \frac{2(s-1)}{(s-1)(s^2-s+6)} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{s+1}{(s-1)(s+2)(s+3)} \\ \frac{-4s}{(s+2)(s+3)(s-2)} \end{pmatrix}$$

Partial fractions:

$$\frac{s+1}{(s-1)(s-2)(s+3)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+3}$$

$$= -\frac{1}{2}\frac{1}{s-1} + \frac{3}{5}\frac{1}{s-2} - \frac{1}{10}\frac{1}{s+3}$$

$$\frac{-4s}{(s+2)(s+3)(s-2)} = \frac{A}{s+2} + \frac{B}{s+3} + \frac{C}{s-2}$$

$$= -\frac{2}{s+2} + \frac{12}{5}\frac{1}{s+3} - \frac{2}{5}\frac{1}{s-2}.$$

$$\vec{x}(t) = \mathcal{L}^{-1}\{\vec{X}(s)\} = \begin{pmatrix} \mathcal{L}^{-1}\left\{-\frac{1}{2}\frac{1}{s-1} + \frac{3}{5}\frac{1}{s-2} - \frac{1}{10}\frac{1}{s+3}\right\} \\ \mathcal{L}^{-1}\left\{-\frac{2}{s+2} + \frac{12}{5}\frac{1}{s+3} - \frac{2}{5}\frac{1}{s-2}\right\} \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} -\frac{1}{2}e^{t} + \frac{3}{5}e^{2t} - \frac{1}{10}e^{-3t} \\ -2e^{-2t} + \frac{12}{5}e^{-3t} - \frac{2}{5}e^{2t} \end{pmatrix}}$$

#20 b) $\vec{x}' = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \vec{x} + \begin{pmatrix} t^{-3} \\ -t^{-2} \end{pmatrix}$

$A = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix}$, $\det A = 0$, $\operatorname{tr} A = 0$.

eigenvalue $\lambda = 0$ (repeated)

corresponding eigenvector $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$\begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 4v_1 - 2v_2 = 0$

$2v_1 = v_2$

$\vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Generalized eigenvector: $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$

$\begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow 4w_1 - 2w_2 = 1$

$w_1 = \dfrac{1+2w_2}{4}$

$\vec{w} = \begin{pmatrix} \frac{1+2w_2}{4} \\ w_2 \end{pmatrix} \overset{w_2=0}{=\!=} \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix}$

General solution of the corresponding homogeneous system:

$\vec{x}_{hom}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \left[t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix} \right]$

Fundamental matrix: $\Psi(t) = \begin{pmatrix} 1 & t + 1/4 \\ 2 & 2t \end{pmatrix}$, $\det \Psi(t) = -1/2$

$\Psi^{-1}(t) = -2 \begin{pmatrix} 2t & -t - 1/4 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -4t & 2t + 1/2 \\ 4 & -2 \end{pmatrix}$

& $\Psi^{-1}(t) \vec{g}(t) = \begin{pmatrix} -4t & 2t + \frac{1}{2} \\ 4 & -2 \end{pmatrix} \begin{pmatrix} \frac{1}{t^3} \\ -\frac{1}{t^2} \end{pmatrix} = \begin{pmatrix} -\frac{4}{t^2} - \frac{2}{t} - \frac{1}{2t^2} \\ \frac{4}{t^3} + \frac{2}{t^2} \end{pmatrix}$

$$\int \Psi^{-1}(t)\vec{g}(t)\,dt = \begin{pmatrix} \int\left(-\frac{9}{2t^2} - \frac{2}{t}\right)dt \\ \int\left(\frac{4}{t^3} + \frac{2}{t^2}\right)dt \end{pmatrix}$$

$$= \begin{pmatrix} \frac{9}{2t} - 2\ln|t| + \cancelto{0}{c_1} \\ -\frac{4}{2t^2} - \frac{2}{t} + \cancelto{0}{c_2} \end{pmatrix} = \begin{pmatrix} \frac{9}{2t} - 2\ln|t| \\ -\frac{2}{t^2} - \frac{2}{t} \end{pmatrix}$$

$$\vec{x}_p(t) = \Psi(t)\int \Psi^{-1}(t)\vec{g}(t)\,dt$$

$$= \begin{pmatrix} 1 & t+1/4 \\ 2 & 2t \end{pmatrix}\begin{pmatrix} \frac{9}{2t} - 2\ln|t| \\ -\frac{2}{t^2} - \frac{2}{t} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{9}{2t} - 2\ln|t| - \left(\frac{2}{t^2} + \frac{2}{t}\right)\left(t + \frac{1}{4}\right) \\ \frac{9}{t} - 4\ln|t| - \frac{4}{t} - 4 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{9}{2t} - 2\ln|t| - \frac{2}{t} - \frac{1}{2t^2} - 2 - \frac{1}{2t} \\ \frac{5}{t} - 4\ln|t| - 4 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{t} - 2\ln|t| - \frac{1}{2t^2} - 2 \\ \frac{5}{t} - 4\ln|t| - 4 \end{pmatrix}$$

$$\vec{x}(t) = \vec{x}_{hom}(t) + \vec{x}_p(t) = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} t+1/4 \\ 2t \end{pmatrix} + \begin{pmatrix} \frac{2}{t} - 2\ln|t| - \frac{1}{2t^2} - 2 \\ \frac{5}{t} - 4\ln|t| - 4 \end{pmatrix}$$

#21. $y''+xy'+2y=0$.

(a) $\cancel{y}=\sum_{n=0}^{\infty} a_n x^n$, $y'=\sum_{n=1}^{\infty} a_n n x^{n-1}$, $y''=\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2}$

Plug into the equation:

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + x\sum_{n=1}^{\infty} a_n n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=1}^{\infty} a_n n x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0.$$

$$a_{0+2}(0+2)(0+1)x^0 + \sum_{n=1}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=1}^{\infty} a_n n x^n$$
$$+ 2a_0 x^0 + \sum_{n=0}^{\infty} 2a_n x^n = 0.$$

$$2a_2 + 2a_0 + \sum_{n=1}^{\infty}\left[a_{n+2}(n+2)(n+1) + na_n + 2a_n\right]x^n = 0.$$

$2a_2+2a_0=0 \Rightarrow a_2=-a_0$

$a_{n+2}(n+2)(n+1)+(n+2)a_n=0 \Rightarrow \boxed{a_{n+2} = -\frac{a_n}{n+1}}\ n\geq 0$

recurrence relation.

a_0

a_1

$a_2=-a_0$

$a_3=-\frac{a_1}{2}$

$a_4=-\frac{a_2}{3}=\frac{a_0}{3}$

$a_5=-\frac{a_3}{4}=\frac{a_1}{2\cdot 4}$

$$a_{2n+1}=(-1)^n\frac{a_1}{2^n\cdot n!}$$

$$a_{2n}=(-1)^n\frac{a_0}{1\cdot 3\cdot\ldots\cdot(2n-1)}$$

$$y(t)=\sum_{n=0}^{\infty}(-1)^n\frac{a_1}{2^n n!}x^{2n+1} + \sum_{n=0}^{\infty}(-1)^n\frac{a_0}{1\cdot 3\cdot\ldots\cdot(2n-1)}x^{2n}$$

$$=a_1\underbrace{\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2^n\cdot n!}}_{y_1(x)} + a_0\underbrace{\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{1\cdot 3\cdot\ldots\cdot(2n-1)}}_{y_2(x)}$$

(b) $y_1(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2^n n!}=x-\frac{x^3}{2}+\frac{x^5}{8}-\frac{x^7}{48}+\dots$

$y_1(0)=0,\quad y_1'(x)=1-\frac{3x^2}{2}+\frac{5x^4}{8}-\frac{7x^6}{48}+\dots$

$y_1'(0)=1$

$y_2(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{1\cdot 3\cdot\ldots\cdot(2n-1)}=1-\frac{x^2}{1}+\frac{x^4}{3}-\frac{x^6}{15}+\dots$

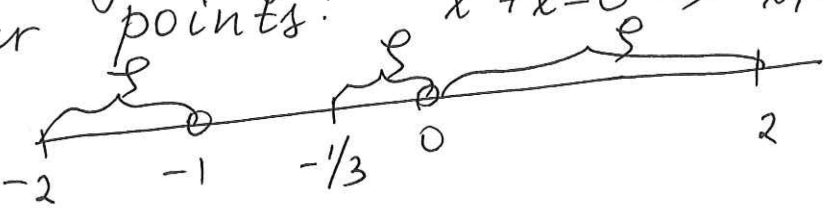
$y_2(0)=1,\quad y_2'(x)=-2x+\frac{4x^3}{3}-\frac{6x^5}{15}+\dots$

$y_2'(0)=0.$

$$W[y_1,y_2](0)=\begin{vmatrix} y_1(0) & y_2(0)\\ y_1'(0) & y_2'(0)\end{vmatrix}=\begin{vmatrix} 0 & 1\\ 1 & 0\end{vmatrix}=-1\neq 0.$$

#22. $(x^2+x)y''+3y'-6xy=0.$

singular points: $x^2+x=0\Rightarrow x_1=0,\ x_2=-1.$



(a) $\rho=1$

(b) $\rho=1/3$

(c) $\rho=2.$

#23. $x^2y''+(1+x)y'+3(\ln x)y=0,\ y(1)=2,\ y'(1)=0.$

$$y''=-\frac{1+x}{x^2}y'-\frac{3\ln x}{x^2}y=-\left(\frac{1}{x^2}+\frac{1}{x}\right)y'-\frac{3\ln x}{x^2}y$$

$$y''(1)=-(1+1)y'(1)-\frac{3\ln 1}{1}y(1)=0$$

$$y'''=(y'')'=\left(\frac{2}{x^3}+\frac{1}{x^2}\right)y'-\left(\frac{1}{x^2}+\frac{1}{x}\right)y''-3y\cdot\frac{x-2x\ln x}{x^4}-\frac{3\ln x}{x^2}y'$$

$$y'''(1)=(2+1)y'(1)-(1+1)y''(1)-3y(1)\frac{1-2\ln 1}{1}-\frac{3\ln 1}{1}y'(1)=\boxed{-3}$$