11. Find a general solution to the equation

$$y'' + 6y' + 9y = \frac{e^{-3x}}{1+2x}$$

variation of parameters.

homogeneous eqn:

$$y''+6y'+9y=0$$
$$r^2+6r+9=0$$
$$(r+3)^2=0$$
$r=-3$ - repeated root

$$y_h(x)=(C_1+C_2x)e^{-3x}$$
$$=C_1\underbrace{e^{-3x}}_{y_1(x)}+C_2\underbrace{xe^{-3x}}_{y_2(x)}$$

$$y(x)=C_1(x)e^{-3x}+C_2(x)xe^{-3x}$$

$$\begin{cases} C_1'y_1+C_2'y_2=0 \\ C_1'y_1'+C_2'y_2'=g(x)\end{cases}$$

$$\begin{cases} C_1'\cancel{e^{-3x}}+C_2'x\cancel{e^{-3x}}=0 \\ C_1'(-3)\cancel{e^{-3x}}+C_2'\cancel{e^{-3x}}+C_2'x(-3)\cancel{e^{-3x}}=\frac{\cancel{e^{-3x}}}{1+2x}\end{cases}$$

$$\begin{cases} C_1'+C_2'x=0 \Rightarrow \boxed{C_1'=-C_2'x} \\ -3C_1'-3xC_2'+C_2'=\frac{1}{1+2x}\end{cases}$$

$$C_2'=\frac{1}{1+2x}$$
$$\boxed{C_2(x)=\frac{1}{2}\ln|1+2x|+C_3}$$

$$C_1'(x)=-xC_2'$$
$$=-\frac{x}{1+2x}$$ - improper fraction

$$\begin{array}{r} \tfrac{1}{2} \\ x\,\overline{)1+2x} \\ \underline{-x+1/2} \\ -1/2 \end{array}$$

$$=-\left(\frac{1}{2}-\frac{1}{2}\frac{1}{1+2x}\right)$$
$$C_1'(x)=-\frac{1}{2}+\frac{1}{2+4x}$$
$$\boxed{C_1(x)=-\frac{1}{2}x+\frac{1}{4}\ln|2+4x|+C_4}$$

$$\boxed{y(x)=\left(-\frac{1}{2}x+\frac{1}{4}\ln|2+4x|+C_4\right)e^{-3x}+\left(\frac{1}{2}\ln|1+2x|+C_3\right)xe^{-3x}}$$

general solution.

A mass weighing 3 lb stretches a spring 3 in. If the mass is pushed upward, contracting the spring a distance of 1 in. then set in motion with a downward velocity of 2 ft/s, and if there is no damping, find the position u of the mass at any time t. Determine the frequency, period and amplitude of the motion.

$$\underbrace{m}_{\text{mass}}u'' + \underbrace{b}_{\text{damping const}}u' + \underbrace{k}_{\text{stiffness}}u = 0 \quad \text{(no external forces)}$$

$b = 0$ (no damping)

$W = 3\,lb$

$W = mg$

$m = \frac{W}{g} = \frac{3}{32}$

Hook's law $\quad W = k\boxed{x}$ elongation

$3 = k \frac{3}{12}$ ft

$k = 12$

IVP:

$$\boxed{\begin{array}{l} \frac{3}{32}u'' + 12u = 0 \\ u(0) = -\frac{1}{12} \\ u'(0) = 2 \end{array}} \qquad \boxed{\begin{array}{l} u'' + 128u = 0 \\ u(0) = -\frac{1}{12} \\ u'(0) = 2 \end{array}}$$

u(t)=C_1 cos(8*sqrt(2)t)+C_2 sin(8*sqrt(2)t),

C_1 =-1/12, C_2 =sqrt(2)/8

A=sqrt(11/2)/12

frequency=4sqrt(2)/pi

period=pi/(4sqrt(2))

3. A mass weighting 8 lb is attached to a spring hanging from the ceiling and comes to rest at its equilibrium position. At $t = 0$, an external force $F(t) = 2\cos 2t$ lb is applied to the system. If the spring constant is 10 lb/ft and the damping constant is 1 lb-sec/ft, find the steady-state solution for the system. What is the resonance frequency for the system?

$$mg = 8 \text{ lb} \Rightarrow m = \frac{8}{32} = \frac{1}{4}$$

$$\gamma = 1, \quad k = 10$$

$$mu'' + \gamma u' + ku = 2\cos 2t$$

$$\frac{1}{4}u'' + u' + 10u = 2\cos 2t$$

$$u'' + u' + 40u = 8\cos 2t$$

<u>steady-state solution:</u>

$$u_{st.s.} = A\cos 2t + B\sin 2t$$

$$A = \frac{18}{85}, \quad B = \frac{4}{85}$$

$$\boxed{u_{st.s.}(t) = \frac{18}{85}\cos 2t + \frac{4}{85}\sin 2t}$$

<u>Resonance frequency</u> $= \dfrac{\sqrt{\frac{k}{m} - \frac{\gamma^2}{2m^2}}}{2\pi}$

$$= \frac{\sqrt{\frac{10}{1/4} - \frac{1}{2\cdot 1/16}}}{2\pi} = \frac{\sqrt{32}}{2\pi} = \boxed{\frac{2\sqrt{2}}{\pi}}$$

1. Find the Laplace transform of the given function.

(a) $f(t) = \begin{cases} \frac{t}{2}, & 0 \leq t < 6 \\ 3, & t \geq 6 \end{cases}$, $f(t) = \frac{t}{2} + \left(3 - \frac{t}{2}\right) u_6(t)$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{\frac{t}{2} + \left(3 - \frac{t}{2}\right) u_6(t)\right\}$$

$$= \mathcal{L}\left\{\frac{t}{2}\right\} + \mathcal{L}\left\{\frac{1}{2}(6-t) u_6(t)\right\}$$

$$= \frac{1}{2s^2} + \frac{1}{2}(-1)\,\mathcal{L}\{(t-6) u_6(t)\}$$

$$= \frac{1}{2s^2} - \frac{1}{2} e^{-6s} \mathcal{L}\{t\} = \boxed{\frac{1}{2s^2} - \frac{e^{-6s}}{2s^2}}$$

(b) $f(t) = (t^2 - 2t + 2)u_1(t)$

$$f(t) = \left[(t-1)^2 + 1\right] u_1(t)$$

$$= (t-1)^2 u_1(t) + u_1(t)$$

$$\mathcal{L}\{(t-1)^2 u_1(t) + u_1(t)\}$$

$$= e^{-s} \mathcal{L}\{t^2\} + \frac{e^{-s}}{s}$$

$$= \boxed{e^{-s}\frac{2}{s^3} + \frac{e^{-s}}{s}}$$

(c) $f(t) = \int_0^t (t-\tau)^2 \cos 2\tau \, d\tau = (g * h)(t)$

$g(t-\tau) = (t-\tau)^2 \Rightarrow g(t) = t^2$

$h(\tau) = \cos 2\tau \Rightarrow h(t) = \cos 2t$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{g(t)\} \cdot \mathcal{L}\{h(t)\}$

$= \mathcal{L}\{t^2\} \cdot \mathcal{L}\{\cos 2t\}$

$= \boxed{\frac{2}{s^3} \cdot \frac{s}{s^2+4}}$

(d) $f(t) = t \cos 3t$

$\mathcal{L}\{t \cos 3t\} = (-1) \frac{d}{ds} \{\mathcal{L}\{\cos 3t\}\}$

$= -\frac{d}{ds}\left(\frac{s}{s^2+9}\right)$

$= -\frac{s^2+9-2s(s)}{(s^2+9)^2}$

$= \boxed{\frac{s^2-9}{(s^2+9)^2}}$

(e) $f(t) = e^t \delta(t-1)$

$\mathcal{L}\{\delta(t-1)\} = e^{-s}$

$\mathcal{L}\{e^t \delta(t-1)\} = \boxed{e^{-(s-1)}}$

2. Find the inverse Laplace transform of the given function.

(a) $F(s) = \dfrac{2s+6}{s^2-4s+8}$

$$\frac{2s+6}{s^2-4s+8} = \frac{2s+6}{(s-2)^2+4} = 2\,\frac{s+3}{(s-2)^2+4}$$

$$= 2\,\frac{s-2+5}{(s-2)^2+4} = 2\frac{s-2}{(s-2)^2+4} + 5\cdot\frac{2}{(s-2)^2+4}$$

$$\mathcal{L}^{-1}\{F(s)\} = 2\,\mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+4}\right\} + 5\,\mathcal{L}^{-1}\left\{\frac{2}{(s-2)^2+4}\right\}$$

$$= \boxed{2e^{2t}\cos 2t + 5e^{2t}\sin 2t}$$

(b) $F(s) = \dfrac{e^{-2s}}{s^2+s-2}$

$$\frac{1}{s^2+s-2} = \frac{1}{(s+2)(s-1)} = \frac{A}{s+2} + \frac{B}{s-1}$$

$$= \frac{A(s-1)+B(s+2)}{(s+2)(s-1)}$$

$$1 = A(s-1)+B(s+2)$$

$s=1$: $\quad 1 = 3B \Rightarrow B = 1/3$

$s=-2$: $\quad 1 = -3A \Rightarrow A = -1/3$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+s-2}\right\} = \mathcal{L}\left\{-\frac{1}{3}\,\frac{1}{s+2} + \frac{1}{3}\,\frac{1}{s-1}\right\}$$

$$= -\frac{1}{3}e^{-2t} + \frac{1}{3}e^{t}$$

$$\mathcal{L}\left\{\frac{e^{-2s}}{s^2+s-2}\right\} = \boxed{\left[-\frac{1}{3}e^{-2(t-2)} + \frac{1}{3}e^{(t-2)}\right]u_2(t)}$$

3. Solve the initial value problem using the Laplace transform:

(a) $y'' + 4y = \begin{cases} t, & 0 \le t < 1 \\ 1, & t \ge 1 \end{cases}$, $y(0) = y'(0) = 0$

$$g(t) = t + (1-t)u_1(t)$$
$$\mathcal{L}\{g(t)\} = \frac{1}{s^2} - \mathcal{L}\{(t-1)u_1(t)\}$$
$$= \frac{1}{s^2} - e^{-s}\mathcal{L}\{t\}$$
$$= \frac{1}{s^2} - e^{-s}\frac{1}{s^2}$$

$$\mathcal{L}\{y(t)\} = Y(s)$$
$$\mathcal{L}\{y'(t)\} = sY(s) - y(0) = sY(s)$$
$$\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s)$$
$$\mathcal{L}\{y''+4y\} = \mathcal{L}\{g(t)\}$$
$$(s^2+4)Y(s) = \frac{1}{s^2} - \frac{e^{-s}}{s^2}$$
$$Y(s) = \frac{1}{s^2(s^2+4)} - \frac{e^{-s}}{s^2(s^2+4)}$$

Partial fractions:

$$\frac{1}{s^2(s^2+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+4}$$
$$= \frac{As(s^2+4) + B(s^2+4) + (Cs+D)s^2}{s^2(s^2+4)}$$

$$1 = s^3(A+C) + s^2(B+D) + s(4A) + 4B$$

s^3: $0 = A + C$

s^2: $0 = B + D$ $\Longrightarrow$ $A = C = 0$, $B = 1/4$, $D = -1/4$

s: $0 = 4A$

1: $1 = 4B$

$$\frac{1}{s^2(s^2+4)} = \frac{1}{4}\frac{1}{s^2} - \frac{1}{4}\frac{1}{s^2+4}$$
$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+4)}\right\} = \frac{1}{4}t - \frac{1}{8}\sin 2t$$
$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2(s^2+4)}\right\} = u_1(t)\left[\frac{1}{4}(t-1) - \frac{1}{8}\sin 2(t-1)\right]$$
$$y(t) = \mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+4)} - \frac{e^{-s}}{s^2(s^2+4)}\right\}$$
$$= \boxed{\frac{1}{4}t - \frac{1}{8}\sin 2t + u_1(t)\left[\frac{1}{4}(t-1) - \frac{1}{8}\sin 2(t-1)\right]}$$

(b) $y'' + 2y' + 3y = \delta(t - 3\pi)$, $y(0) = y'(0) = 0$

$$\mathcal{L}\{y'' + 2y' + 3y\} = \mathcal{L}\{\delta(t-3\pi)\}$$

$$\mathcal{L}\{y\} = Y(s)$$

$$\mathcal{L}\{y'\} = sY(s) - y(0)$$
$$= sY(s)$$

$$\mathcal{L}\{y''\} = s^2 Y(s) - s\,y(0) - y'(0)$$
$$= s^2 Y(s)$$

$$\mathcal{L}\{\delta(t-3\pi)\} = e^{-3\pi s}$$

$$(s^2 + 2s + 3)\,Y(s) = e^{-3\pi s}$$

$$Y(s) = \frac{1}{s^2+2s+3}\,e^{-3\pi s}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{e^{-3\pi s}}{s^2+2s+3}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+3}\right\}$$
$$= \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+2}\right\}$$
$$= \frac{1}{\sqrt{2}}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{(s+1)^2+2}\right\}$$
$$= \frac{1}{\sqrt{2}} e^{-t} \sin\sqrt{2}\,t$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-3\pi s}}{s^2+2s+3}\right\}$$
$$= \boxed{u_{3\pi}(t)\,\frac{1}{\sqrt{2}}\,e^{-(t-3\pi)} \sin\sqrt{2}(t-3\pi) = y(t)}$$

(c) $y'' + 4y' + 4y = g(t)$, $y(0) = 2$, $y'(0) = -3$

$$\mathcal{L}\{y'' + 4y' + 4y\} = \mathcal{L}\{g(t)\}$$
$$\mathcal{L}\{g(t)\} = G(s)$$
$$\mathcal{L}\{y\} = Y(s)$$
$$\mathcal{L}\{y'\} = sY(s) - y(0) = sY(s) - 2$$
$$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0)$$
$$= s^2Y(s) - 2s + 3$$
$$s^2Y(s) - 2s + 3 + 4sY(s) - 8 + 4Y(s) = G(s)$$
$$Y(s)(s^2+4s+4) = G(s) + 2s + 5$$
$$Y(s) = \frac{G(s)}{s^2+4s+4} + \frac{2s+5}{s^2+4s+4}$$
$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{G(s)}{s^2+4s+4} + \frac{2s+5}{s^2+4s+4}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+4s+4}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\}$$
$$= e^{-2t}t$$
$$\frac{2s+5}{s^2+4s+4} = \frac{A}{s+2} + \frac{B}{(s+2)^2}$$
$$= \frac{A(s+2)+B}{(s+2)^2}$$

$$2s+5 = A(s+2) + B$$
$s=-2$: $1 = B$

$s=0$: $5 = 2A + B$, $2A = 5 - B = 4$

$A = 2$

$$\frac{2s+5}{s^2+4s+4} = \frac{2}{s+2} + \frac{1}{(s+2)^2}$$
$$\mathcal{L}^{-1}\left\{\frac{2s+5}{s^2+4s+4}\right\} = 2e^{-2t} + e^{-2t}t$$
$$\boxed{y(t) = \int_0^t g(t-\tau)e^{-2\tau}\tau\,d\tau + 2e^{-2t} + e^{-2t}t}$$

4. Find A^{-1} if $A = \begin{pmatrix} 1+i & -1+2i \\ 3+2i & 2-i \end{pmatrix}$

$$\det A = (1+i)(2-i) - (-1+2i)(3+2i) = 2-i+2i-\underbrace{i^2}_{-1}+3+2i-6i-4\underbrace{i^2}_{-1}$$

$$= 5+5-3i = 10-3i$$

$$\frac{1}{\det A} = \frac{1}{10-3i} = \frac{10+3i}{(10-3i)(10+3i)} = \frac{10+3i}{100-9i^2} = \frac{10+3i}{109}$$

$$\boxed{A^{-1} = \frac{10+3i}{109}\begin{pmatrix} 2-i & 1-2i \\ -3-2i & 1+i \end{pmatrix}}$$

5. Find BA if $A = \begin{pmatrix} 1+i & -1+2i \\ 3+2i & 2-i \end{pmatrix}$, $B = \begin{pmatrix} i & 3 \\ 2 & -2i \end{pmatrix}$

$$BA = \begin{pmatrix} i & 3 \\ 2 & -2i \end{pmatrix}\begin{pmatrix} 1+i & -1+2i \\ 3+2i & 2-i \end{pmatrix} = \begin{pmatrix} i(1+i)+3(3+2i) & i(-1+2i)+3(2-i) \\ 2(1+i)-2i(3+2i) & 2(-1+2i)-2i(2-i) \end{pmatrix}$$

$$= \begin{pmatrix} i+i^2+9+6i & -i+2i^2+6-3i \\ 2+2i-6i-4i^2 & -2+4i-4i+4i^2 \end{pmatrix} = \boxed{\begin{pmatrix} 8+7i & 4-4i \\ 6-4i & -6 \end{pmatrix}}$$

9. Find all eigenvalues and eigenvectors of the martix

$$\begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$$

$$A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix}$$

$$= (-\lambda)(3-\lambda)^2 + 16 + 16 + 16\lambda - 4(3-\lambda) - 4(3-\lambda)$$

$$= -\lambda^3 + 6\lambda^2 + 15\lambda + 8 = 0$$

$$\lambda^3 - 6\lambda^2 - 15\lambda - 8 = 0$$

plug $\lambda = -1$: $\quad -1 - 6 + 15 - 8 = 0$

one eigenvalue is $\lambda_1 = -1$.

divide by long divisions:

$$\begin{array}{r|l} & \lambda^2 - 7\lambda + 8 \\ \lambda + 1 & \lambda^3 - 6\lambda^2 - 15\lambda - 8 \\ & \underline{\lambda^3 + \lambda^2} \\ & -7\lambda^2 - 15\lambda \\ & \underline{-7\lambda^2 - 7\lambda} \\ & -8\lambda - 8 \\ & \underline{-8\lambda - 8} \\ & 0. \end{array}$$

$$\lambda^3 - 6\lambda^2 - 15\lambda - 8 = (\lambda+1)(\lambda^2 - 7\lambda + 8) = (\lambda+1)^2(\lambda - 8)$$

Eigenvalues : $\lambda_1 = -1$ — repeated

$\lambda_2 = 8$

corresponding eigenvectors:

$\lambda_1 = -1$

$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ is a solution to

$(A+I)\vec{v} = \vec{0}$

$$\begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 4v_1 + 2v_2 + 4v_3 = 0 \\ 2v_1 + v_2 + 2v_3 = 0 \\ 4v_1 + 2v_2 + 4v_3 = 0 \end{cases} \Longrightarrow v_2 = 2v_3 + 2v_1$$

$$\vec{v} = \begin{pmatrix} v_1 \\ 2v_1 + 2v_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 \\ 2v_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2v_3 \\ v_3 \end{pmatrix}$$

$$= v_1 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + v_3 \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}.$$

if we plug $v_1 = 0$ and $v_3 = 1$, then we'll get $\cancel{\vec{v}_1} =$ the vector $\vec{x}_1 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

if we plug $v_1 = 1$ and $v_3 = 1$, then we'll get the vector $\vec{x}_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$.

the eigenvalue $\lambda_1 = 1$ has <u>two</u> corresponding eigenvectors: $\vec{x}_1 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

and $\vec{x}_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

$\lambda_2 = 8.$

$\vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ is a solution of

$(A - 8I)\vec{w} = \vec{0}$

$$\begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -5w_1 + 2w_2 + 4w_3 = 0 \\ w_1 - 4w_2 + w_3 = 0 \\ 4w_1 + 2w_2 - 5w_3 = 0 \end{cases}$$

Solve the 2nd equation for w_3

$$w_3 = 4w_2 - w_1$$

and then plug into the 1st and the 3rd

$$\begin{cases} -9w_1 + 18w_2 = 0 \\ 9w_1 - 18w_2 = 0 \end{cases} \Rightarrow \quad w_1 = 2w_2$$

and $\quad w_3 = 4w_2 - 2w_2 = 2w_2.$

$$\vec{w} = \begin{pmatrix} 2w_2 \\ w_2 \\ 2w_2 \end{pmatrix} = w_2 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

$\boxed{\vec{x}_3 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \text{ eigenvector corresponding to } \lambda_2 = 8}$

6. Find the general solution of the system

(a) $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\mathbf{x}$ $A = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}$, $A - \lambda I = \begin{pmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{pmatrix}$

Eigenvalues: $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix} = (1-\lambda)(-2-\lambda) - 4 = 0$$

$$-2 - \lambda + 2\lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 + \lambda - 6 = 0$$

$$(\lambda+3)(\lambda-2) = 0$$

$\lambda_1 = -3$, $\lambda_2 = 2$ – eigenvalues

Corresponding eigenvectors:

$\lambda_1 = -3$. Corresponding eigenvector $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is a solution of the system $(A + 3I)\vec{v} = \vec{0}$

$$\begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$4v_1 + v_2 = 0 \Rightarrow v_2 = -4v_1$$

$$\vec{v} = \begin{pmatrix} v_1 \\ -4v_1 \end{pmatrix} \overset{v_1=1}{=} \boxed{\begin{pmatrix} 1 \\ -4 \end{pmatrix} \text{ corresponds to } \lambda_1 = -3}$$

$\lambda_2 = 2$. $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$

$$(A - 2I)\vec{w} = \vec{0}$$

$$\begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-w_1 + w_2 = 0 \Rightarrow w_1 = w_2$$

$$\vec{w} = \begin{pmatrix} w_1 \\ w_1 \end{pmatrix} \overset{w_1=1}{=} \boxed{\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ corresponds to } \lambda_2 = 2}$$

General solution:

$$\boxed{\vec{x}(t) = C_1\begin{pmatrix} 1 \\ -4 \end{pmatrix}e^{-3t} + C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{2t}}$$

$$\vec{x}' = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix} \vec{x}, \quad A = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix}, \quad A - \lambda I = \begin{pmatrix} -3-\lambda & 2 \\ -1 & -1-\lambda \end{pmatrix}$$

eigenvalues:

$$\begin{vmatrix} -3-\lambda & 2 \\ -1 & -1-\lambda \end{vmatrix} = (3+\lambda)(1+\lambda) + 2$$

$$= 3 + 4\lambda + \lambda^2 + 2 = 0$$

$$\lambda^2 + 4\lambda + 5 = 0$$

$$\lambda_1 = \frac{-4 + \sqrt{16-20}}{2} = -2 + i, \quad \lambda_2 = -2 - i$$

Corresponding eigenvector: $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$,

$$(A - (-2+i) I)\vec{v} = \vec{0}$$

$$\begin{pmatrix} -3-(-2+i) & 2 \\ -1 & -1-(-2+i) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1-i & 2 \\ -1 & 1-i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -(1+i)v_1 + 2v_2 = 0 \\ -v_1 + (1-i)v_2 = 0 \end{cases} \Longrightarrow \quad v_1 = (1-i)v_2$$

$$\vec{v} = \begin{pmatrix} (1-i)v_2 \\ v_2 \end{pmatrix} \overset{v_2 = 1}{=\!=} \begin{pmatrix} 1-i \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

solutions:

$$\vec{v} e^{\lambda_1 t} = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right] e^{(-2+i)t} \qquad \boxed{e^{(-2+i)t} = e^{-2t}(\cos t + i \sin t)}$$

$$= \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right] e^{-2t}(\cos t + i \sin t)$$

$$= e^{-2t} \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos t + i \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin t + i \begin{pmatrix} -1 \\ 0 \end{pmatrix} \cos t + i^2 \begin{pmatrix} -1 \\ 0 \end{pmatrix} \sin t \right]$$

$$= e^{-2t} \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos t - \begin{pmatrix} -1 \\ 0 \end{pmatrix} \sin t + i \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin t + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \cos t \right] \right\}$$

$$= e^{-2t} \left[\begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix} + i \begin{pmatrix} \sin t - \cos t \\ \sin t \end{pmatrix} \right]$$

General solution $\boxed{\vec{x}(t) = \left[c_1 \begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix} + c_2 \begin{pmatrix} \sin t - \cos t \\ \sin t \end{pmatrix} \right] e^{-2t}}$