# Chapter 4. Linear Second Order Equations

## Section 4.2 **Linear Differential Operators**

A **linear second order equation** is an equation that can be written in the form

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = b(x). \tag{1}$$

We will assume that $a_0(x)$, $a_1(x)$, $a_2(x)$, $b(x)$ are continuous functions of $x$ on an interval $I$. When $a_0$, $a_1$, $a_2$, $b$ are constants, we say the equation has **constant coefficients**, otherwise it has **variable coefficients**.

For now, we are interested in those linear equations for which $a_2(x)$ is never zero on $I$. In that case we can rewrite (1) in the **standard form**

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = g(x), \tag{2}$$

where $p(x) = a_1(x)/a_2(x)$, $q(x) = a_0(x)/a_2(x)$ and $g(x) = b(x)/a_2(x)$ are continuous on $I$.

Associated with equation (2) is the equation

$$y'' + p(x)y' + q(x)y = 0, \tag{3}$$

which is obtained from (2) by replacing $g(x)$ with zero. We say that equation (2) is a **nonhomogeneous equation** and that (3) is the corresponding **homogeneous equation**.

$$L[y] = y''(x) + p(x)y'(x) + q(x)y(x). \tag{4}$$

**Lemma 1.** Let $L[x] = y''(x) + p(x)y'(x) + q(x)y(x)$. If $y$, $y_1$, and $y_2$ are any twice-differentiable functions on the interval $I$ and if $c$ is any constant, then

$$L[y_1 + y_2] = L[y_1] + L[y_2], \tag{5}$$

$$L[cy] = cL[y]. \tag{6}$$

**Theorem 1 (linear combination of solutions).** Let $y_1$ and $y_2$ be solutions to the *homogeneous* equation (3). Then any linear combination $C_1y_1 + C_2y_2$ of $y_1$ and $y_2$, where $C_1$ and $C_2$ are constants, is also the solution to (3).

There are *basic differentiation operators* with respect to $x$:

$$Dy = \frac{dy}{dx}, \quad D^2y = \frac{d^2y}{dx^2}, \ldots, D^ny = \frac{d^ny}{dx^n}.$$

Using these operators we can express $L$ defined in (4) as

$$L[y] = D^2y + pDy + qy = (D^2 + pD + q)y.$$

When $p$ and $q$ are *constants*, we can even treat $D^2 + pD + q$ as a polynomial in $D$ and factor it.

**Example 1.** Express the operator

$$x^2y'' - xy' + y$$

using the differential operator $D$.

**Theorem 2 (existence and uniqueness of solution).** Suppose $p(x)$, $q(x)$, and $g(x)$ are continuous on some interval $(a, b)$ that contains the point $x_0$. Then, for any choice of initial values $y_0$, $y_1$ there exists a unique solution $y(x)$ on the whole interval $(a, b)$ to the initial value problem

$$y'' + p(x)y' + q(x)y = g(x),$$
$$y(x_0) = y_0, \quad y'(0) = y_1.$$

**Example 2.** Find the largest interval for which Theorem 2 ensures the existence and uniqueness of solution to the initial value problem

$$\mathrm{e}^x y'' - \frac{y'}{x-3} + y = \ln x,$$

$$y(1) = y_0, \qquad y'(1) = y_1,$$

where $y_0$ and $y_1$ are real constants.

## Section 4.3. **Fundamental solutions of homogeneous equations**

Theorem 3. Let $y_1$ and $y_2$ denote two solutions on $I$ to

$$y'' + p(x)y' + q(x)y = 0, \tag{7}$$

where $p(x)$ and $q(x)$ are continuous on $I$. Suppose at some point $x_0 \in I$ these solutions satisfy

$$y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0) \neq 0. \tag{8}$$

Then every solution to (7) on $I$ can be expressed in the form

$$y(x) = C_1y_1(x) + C_2y_2(x), \tag{9}$$

where $C_1$ and $C_2$ are constants.

**Definition 1.** For any two differentiable functions $y_1$ and $y_2$, the determinant

$$W[y_1, y_2](x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_1'(x)y_2(x)$$

is called the **Wronskian** of $y_1$ and $y_2$.

**Definition 2.** A pair of solutions $\{y_1, y_2\}$ to $y'' + p(x)y' + q(x)y = 0$ on $I$ is called **fundamental solution set** if

$$W[y_1, y_2](x_0) \neq 0$$

at some $x_0 \in I$.

**Procedure for solving homogeneous equations**

To determine all solutions to $y'' + p(x)y' + q(x)y = 0$:

(a) Find two solutions $y_1$ and $y_2$ that constitute a fundamental solution set.

(b) Form the linear combination

$$y(x) = C_1 y_1(x) + C_2 y_2(x),$$

to obtain the general solution.

**Definition 3.** Two functions $y_1$ and $y_2$ are said to be **linearly dependent on I** if there exist constants $C_1$ and $C_2$, not both zero, such that

$$C_1 y_1(x) + C_2 y_2(x) = 0$$

for all $x \in I$. If two functions are not linearly dependent, they are said to be **linearly independent**.

**Example 3.** Determine whether the following pairs of functions $y_1$ and $y_2$ are linearly dependent on $[-3, 3]$.

(a) $y_1(x) = \mathrm{e}^{-x} \cos 2x$, $y_1(x) = \mathrm{e}^{-x} \sin 2x$.

(b) $y_1(x) = \sin 2x$, $y_2(x) = \sin x \cos x$.

(c) $y_1(x) = x$, $y_2(x) = |x|$.

**Theorem 4.** Let $y_1$ and $y_2$ be solutions to the equation $y'' + p(x)y' + q(x)y = 0$ on $I$, and let $x_0 \in I$. Then $y_1$ and $y_2$ are linearly dependent on $I$ if and only if the constant vectors

$$\begin{pmatrix} y_1(x_0) \\ y_1'(x_0) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} y_2(x_0) \\ y_2'(x_0) \end{pmatrix}$$

are linearly dependent.

**Corollary 1.** If $y_1$ and $y_2$ are solutions to $y'' + p(x)y' + q(x)y = 0$ on $I$, then the following statements are equivalent:

(i) $\{y_1, y_2\}$ is a fundamental solution set on $I$.

(ii) $y_1$ and $y_2$ are linearly independent on $I$.

(iii) $W[y_1, y_2]$ is never zero on $I$.

**Example 4.** Show that $y_1(x) = x^2$ and $y_2(x) = \frac{1}{x}$ are solutions to

$$x^2 y'' - 2y = 0$$

on the interval $(0, +\infty)$ and give a general solutions.