Chapter 4. Linear Second Order Equations

Section 4.4 **Reduction of Order**

A general solution to a linear second order homogeneous equation is given by a linear combination of two linearly independent solutions.

Let f be nontrivial solution to equation

$$y'' + p(x)y' + q(x)y = 0.$$

Let's try to find solution of the form

$$y(x) = v(x)f(x),$$

where $v(x)$ is an unknown function. Differentiating, we have

$$y' = v'f + vf',$$

$$y'' = v''f + 2v'f' + vf''.$$

Substituting these expression into equation gives

$$v''f + 2v'f' + vf'' + p(v'f + vf') + qvf = 0$$

or

$$(f'' + pf' + qf)v + fv'' + (2f' + pf)v' = 0.$$

Since f is the solution,

$$fv'' + (2f' + pf)v' = 0.$$

Let's $w(x) = v'(x)$, then we have

$$fw' + (2f' + pf)w = 0,$$

separating the variables and integrating gives

$$\frac{dw}{w} = (-2\frac{f'}{f} - p)dx,$$

$$\int \frac{dw}{w} = -2\int \frac{f'}{f}dx - \int pdx,$$

$$\ln|w| = \ln|f^{-2}| - \int pdx,$$

$$w = \pm\frac{e^{\int pdx}}{f^2},$$

which holds on any interval where $f(x) \neq 0$.

$$v' = \pm \frac{e^{\int p dx}}{f^2},$$

$$v = \pm \int \frac{e^{\int p(x) dx}}{[f(x)]^2}.$$

Examples.

(a) Given that $f(x) = e^{3x}$ is a solution to

$$y'' + 2y' - 15y = 0,$$

determine the second linear independent solution.

SOLUTION Let $y(x) = v(x)f(x) = v(x)e^{3x}$, then

$$y'(x) = (v' + 3v)e^{3x},$$

$$y''(x) = (v'' + 6v' + 9v)e^{3x}.$$

Substituting these representations into equation gives

$$(v'' + 6v' + 9v)e^{3x} + 2(v' + 3v)e^{3x} - 15ve^{3x} = 0,$$

which simplifies to

$$v'' + 8v' = 0.$$

Let $w(x) = v'(x)$, then $w'(x) = v''(x)$, $w' = \frac{dw}{dx}$ and

$$\frac{dw}{dx} + 8w = 0.$$

Separation the variables and integrating gives

$$\frac{dw}{w} = -8dx,$$

$$\int \frac{dw}{w} = -\int 8dx,$$

$$\ln|w| = -8x + C,$$

$$w = C_1 e^{-8x},$$

where $C_1 = e^C$.
Since $w(x) = v'(x)$,

$$v' = w = C_1 e^{-8x},$$

$$v = -\frac{C_1}{8} e^{-8x} + C_2 = C_3 e^{-8x} + C_2.$$

Then the general solution to the given equation is
$y(x) = (C_3 e^{-8x} + C_2)e^{3x} = C_3 e^{-5x} + C_2 e^{3x}$.

(b) Given that $f(x) = \frac{1}{x}$ is a solution to

$$x^2 y'' - 2xy' - 4y = 0, \qquad x > 0$$

determine the second linear independent solution.

SOLUTION Let $y(x) = v(x)f(x) = \frac{v(x)}{x}$, then

$$y' = \frac{v'x - v}{x^2} = \frac{v'}{x} - \frac{v}{x^2},$$

$$y'' = \frac{v''x - v'}{x^2} - \frac{v'x^2 - 2xv}{x^4} = \frac{v''}{x} - 2\frac{v'}{x^2} - 2\frac{v}{x^3}.$$

Substituting these representations into equation gives

$$x^2(\frac{v''}{x} - 2\frac{v'}{x^2} - 2\frac{v}{x^3}) - 2x(\frac{v'}{x} - \frac{v}{x^2}) - 4\frac{v(x)}{x} = 0,$$

$$xv'' - 4v' = 0.$$

Let $w(x) = v'(x)$, then $w'(x) = v''(x)$, $w' = \frac{dw}{dx}$ and

$$xw' - 4w = 0.$$

Separation the variables and integrating gives

$$\frac{dw}{w} = 4\frac{dx}{x},$$

$$\int \frac{dw}{w} = 4\int \frac{dx}{x},$$

$$w = Cx^4,$$

$$v = C\frac{x^5}{5} + C_1 = C_2 x^5 + C_1.$$

Then the general solution to the given equation is

$$y(x) = (C_2 x^5 + C_1)\frac{1}{x} = C_2 x^4 + C_1\frac{1}{x}.$$

(c) The equation

$$xy''' - xy'' + y' - y = 0$$

has $f(x) = e^x$ as a solution. Use the substitution $y(x) = v(x)f(x)$ to reduce this third order equation to a second order equation.

SOLUTION Let $y(x) = v(x)f(x) = v(x)e^x$, then

$$y'(x) = (v' + v)\mathrm{e}^x,$$

$$y''(x) = (v'' + 2v' + v)\mathrm{e}^x,$$

$$y'''(x) = (v''' + 3v'' + 3v' + v)\mathrm{e}^x.$$

Substituting these representations into equation gives

$$x(v''' + 3v'' + 3v' + v)\mathrm{e}^x - x(v'' + 2v' + v)\mathrm{e}^x + (v' + v)\mathrm{e}^x - v\mathrm{e}^x = 0,$$

$$xv''' + 2xv'' + (x + 1)v' = 0.$$

Let $w(x) = v'(x)$, then for w we will have the second order linear differential equation

$$xw'' + 2xw' + (x + 1)w = 0.$$