Chapter 4. Linear Second Order Equations

Section 4.5 **Homogeneous Linear Equations with Constant Coefficients**

For the equation

$$ay'' + by' + cy = 0, \tag{1}$$

where a, b, c are constants, we try to find a solution of the form $y = \mathrm{e}^{rx}$. If we substitute $y = \mathrm{e}^{rx}$ into (1), we obtain

$$(ar^2 + br + c)\mathrm{e}^{rx} = 0.$$

Since e^{rx} is never zero,

$$ar^2 + br + c = 0. \tag{2}$$

Consequently, $y = \mathrm{e}^{rx}$ is a solution to (1) if an only if r satisfies (2). Equation (2) is called the **auxiliary equation** or **characteristic equation** associated with equation (1).

So, the equation (2) is a quadratic, and its roots are:

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \qquad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

When $\mathbf{\sqrt{b^2 - 4ac} > 0}$, then $r_1, r_2 \in \mathbf{R}$ and $r_1 \neq r_2$. So, $y_1(x) = \mathrm{e}^{r_1 x}$ and $y_2(x) = \mathrm{e}^{r_2 x}$ are two linearly independent solutions to (1) and

$$y(x) = c_1 \mathrm{e}^{r_1 x} + c_2 \mathrm{e}^{r_2 x}$$

is the general solution to (1).

If $\mathbf{\sqrt{b^2 - 4ac} = 0}$, then the equation (2) has a repeated root $r \in \mathbf{R}$, $r = -\frac{b}{2a}$. In this case, $y_1(x) = \mathrm{e}^{rx}$ and $y_2(x) = x\mathrm{e}^{rx}$ are two linearly independent solutions to (1) and

$$y(x) = c_1 \mathrm{e}^{rx} + c_2 x \mathrm{e}^{rx} = (c_1 + c_2 x)\mathrm{e}^{rx}$$

is the general solution to (1).

Example 1. Find a general solution to the given equation

(a) $y'' - y' - 2y = 0.$

SOLUTION The associated auxiliary equation is

$$r^2 - r - 2 = 0,$$

which has two roots

$$r_1 = \frac{1 + \sqrt{1 + 8}}{2} = 2, \qquad r_2 = \frac{1 - \sqrt{1 + 8}}{2} = -1.$$

Thus, $\{\mathrm{e}^{2x}, \mathrm{e}^{-x}\}$ is a fundamental solution set, and a general solution is

$$y(x) = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-x}.$$

(b) $y'' + 6y' + 9y = 0$.
SOLUTION The associated auxiliary equation is

$$r^2 + 6r + 9 = (r+3)^2 = 0,$$

which has one repeated root $r = -3$.
Thus, $\{e^{-3x}, xe^{-3x}\}$ is a fundamental solution set, and a general solution is

$$y(x) = c_1 e^{-3x} + c_2 x e^{-3x} = (c_1 + c_2 x)e^{-3x}.$$

(c) $y'' - 5y' + 6y = 0$.
SOLUTION The associated auxiliary equation is

$$r^2 - 5r + 6 = 0,$$

which has two roots $r_1 = 2$, $r_2 = 3$.
Thus, $\{e^{2x}, e^{3x}\}$ is a fundamental solution set, and a general solution is

$$y(x) = c_1 e^{2x} + c_2 e^{3x}.$$

(d) $3y'' + 11y' - 7y = 0$.
SOLUTION The associated auxiliary equation is

$$3r^2 + 11r - 7 = 0,$$

which has two roots

$$r_1 = \frac{-11 + \sqrt{121 + 84}}{6} = \frac{-11 + \sqrt{205}}{6},$$

$$r_1 = \frac{-11 - \sqrt{205}}{6}.$$

Thus, $\{e^{\frac{-11+\sqrt{205}}{6}x}, e^{\frac{-11-\sqrt{205}}{6}x}\}$ is a fundamental solution set, and a general solution is

$$y(x) = c_1 e^{\frac{-11+\sqrt{205}}{6}x} + c_2 e^{\frac{-11-\sqrt{205}}{6}x}.$$

Example 2. Solve the given initial value problems.

(a) $y'' + y' = 0$, $y(0) = 2$, $y'(0) = 1$.
SOLUTION The associated auxiliary equation is

$$r^2 + r = r(r+1) = 0,$$

which has two roots $r_1 = 0$, $r_2 = -1$. Then $\{1, e^{-x}\}$ is a fundamental solution set, and a general solution is

$$y(x) = c_1 + c_2 e^{-x}.$$

To find the specific solution that satisfies the initial conditions, we have to plug $x = 0$ into $y(x)$ and $y'(x)$.

$$y(0) = c_1 + c_2 = 2,$$

$$y'(x) = -c_2 e^{-x},$$

$$y'(0) = -c_2 = 1.$$

So, $c_2 = -1$, $c_1 = 2 - c_2 = 2 - (-1) = 3$ and

$$y(x) = 3 - e^{-x}$$

is the solution to the given initial value problem.

(b) $y'' - 4y' + 4y = 0$, $y(1) = 1$, $y'(1) = 1$.
SOLUTION The associated auxiliary equation is

$$r^2 - 4r + 4 = (r-2)^2 = 0,$$

which has one repeated root $r = 2$. Then $\{e^{2x}, xe^{2x}\}$ is a fundamental solution set, and a general solution is

$$y(x) = (c_1 + c_2 x)e^{2x}.$$

To find the specific solution that satisfies the initial conditions, we have to plug $x = 1$ into $y(x)$ and $y'(x)$.

$$y(1) = (c_1 + c_2)e^2 = 1,$$

$$y'(x) = (c_2 + 2c_1 + 2c_2 x)e^{2x},$$

$$y'(1) = (c_2 + 2c_1 + 2c_2)e^2 = (2c_1 + 3c_2)e^2 = 1.$$

So, we have system

$$\begin{cases} c_1 + c_2 = e^{-2}, \\ 2c_1 + 3c_2 = e^{-2}. \end{cases}$$

Multiplying the first equation by (-2) and after that adding two equations gives $c_2 = -e^{-2}$. Since $c_1 = e^{-2} - c_2$, $c_1 = e^{-2} + e^{-2} = 2e^{-2}$. Thus

$$y(x) = (2e^{-2} - xe^{-2})e^{2x} = (2-x)e^{2x-2}$$

is the solution to the given initial value problem.

Cauchy-Euler Equations

A linear second order differential equation that can be expressed in the form

$$ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0, \tag{3}$$

where a, b, and c are constants is called a **homogeneous Cauchy-Euler equation**.

To solve a homogeneous Cauchy-Euler equation, we make the substitution $x(t) = \mathrm{e}^t$. Because $x(t) = \mathrm{e}^t$, it follows by the Chain Rule that

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dx}\mathrm{e}^t = x\frac{dy}{dx},$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(x\frac{dy}{dx}\right) = \frac{dx}{dt}\frac{dy}{dx} + x\frac{d}{dt}\left(\frac{dy}{dx}\right) = \mathrm{e}^t\frac{dy}{dx} + x\frac{d^2y}{dx^2}\frac{dx}{dt} =$$

$$\mathrm{e}^t\frac{dy}{dx} + x\mathrm{e}^t\frac{d^2y}{dx^2} = x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2} = \frac{dy}{dt} + x^2\frac{d^2y}{dx^2}.$$

Thus,

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}.$$

Substituting into (3) the expressions for $x\frac{dy}{dx}$ and $x^2\frac{d^2y}{dx^2}$ gives

$$a\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + b\frac{dy}{dt} + cy = 0,$$

$$a\frac{d^2y}{dt^2} + (b-a)\frac{dy}{dt} + cy = 0,$$

which is linear second order differential equation with constant coefficients.

Example 3. Find the general solution to the equation

$$x^2y''(x) + 7xy'(x) - 7y(x) = 0$$

for $x > 0$.

SOLUTION In this case $a = 1$, $b = 7$, $c = 7$. Substitution $x = \mathrm{e}^t$ transforms given equation into an equation

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} - 7y = 0.$$

The associated auxiliary equation is

$$r^2 + 6r - 7 = 0,$$

which has two roots $r_1 = 1$, $r_2 = -7$. Then $\{\mathrm{e}^x, \mathrm{e}^{-7x}\}$ is a fundamental solution set, and a general solution is

$$y(t) = c_1\mathrm{e}^t + c_2\mathrm{e}^{-7t} = c_1\mathrm{e}^t + c_2(\mathrm{e}^t)^{-7}.$$

Expressing y in terms of the original variable x, we find

$$y(x) = c_1x + c_2x^{-7}.$$