Chapter 4. Linear Second Order Equations

$$ay'' + by' + cy = 0, \tag{1}$$

where a, b, c are constants. The associated auxiliary equation is

$$ar^2 + br + c = 0. \tag{2}$$

Consequently, $y = e^{rx}$ is a solution to (1) if an only if r satisfies (2).
So, the equation (2) is a quadratic, and its roots are:

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \qquad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

When $\sqrt{\mathbf{b^2 - 4ac}} > \mathbf{0}$, then $r_1, r_2 \in \mathbf{R}$ and $r_1 \neq r_2$. So, $y_1(x) = e^{r_1 x}$ and $y_2(x) = e^{r_2 x}$ are two linearly independent solutions to (1) and

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

is the general solution to (1).

If $\sqrt{\mathbf{b^2 - 4ac}} = \mathbf{0}$, then the equation (2) has a repeated root $r \in \mathbf{R}$, $r = -\frac{b}{2a}$. In this case, $y_1(x) = e^{rx}$ and $y_2(x) = xe^{rx}$ are two linearly independent solutions to (1) and

$$y(x) = c_1 e^{rx} + c_2 x e^{rx} = (c_1 + c_2 x)e^{rx}$$

is the general solution to (1).

Example 1.
(a) Find the general solution to the equation

$$y'' - 2y' - 8y = 0.$$

SOLUTION The associated auxiliary equation is

$$r^2 - 2r - 8 = 0,$$

which roots are $r_1 = 4$, $r_2 = -2$, the fundamental solution set is $\{e^{4x}, e^{-2x}\}$. Thus, the general solution is

$$y(x) = c_1 e^{4x} + c_2 e^{-2x}.$$

(b) Solve the given initial value problem $4y'' - 12y' + 9y = 0$, $y(0) = 1$, $y'(0) = \frac{3}{2}$.

SOLUTION The associated characteristic equation is

$$4r^2 - 12r + 9 = (2r - 3)^2 = 0,$$

which has one repeated root $r = \frac{3}{2}$. The fundamental solution set is $\{e^{\frac{3}{2}x}, xe^{\frac{3}{2}x}\}$
So, the general solution to the given equation is

$$y(x) = c_1 e^{\frac{3}{2}x} + c_2 x e^{\frac{3}{2}x} = (c_1 + c_2 x)e^{\frac{3}{2}x}.$$

To find the solution to the initial value problem we have to plug $x = 0$ into $y(x)$ and $y'(x)$.

$$y'(x) = \frac{3}{2}(c_1 + c_2 x)e^{\frac{3}{2}x} + c_2 e^{\frac{3}{2}x} = (\frac{3}{2}c_1 + c_2 + \frac{3}{2}c_2 x)e^{\frac{3}{2}x},$$

$$y(0) = c_1 = 1,$$
$$y'(0) = \frac{3}{2}c_1 + c_2 = \frac{3}{2}.$$

Since $c_1 = 1$ and $c_2 = \frac{3}{2} - \frac{3}{2}c_1 = 0$, the solution to the initial value problem is

$$y(x) = e^{\frac{3}{2}x}.$$

Section 4.6 **Auxiliary Equation with Complex Roots**

If $\sqrt{\mathbf{b^2 - 4ac}} < \mathbf{0}$, then the equation (2) has two complex conjugate roots

$$r_1 = -\frac{b}{2a} + i\frac{\sqrt{4ac - b^2}}{2a} = \alpha + i\beta,$$
$$r_2 = -\frac{b}{2a} - i\frac{\sqrt{4ac - b^2}}{2a} = \alpha - i\beta = \bar{r_1},$$

here $i^2 = -1$, $\alpha = -\frac{b}{2a}$, $\beta = \frac{\sqrt{4ac-b^2}}{2a}$, $\alpha, \beta \in \mathbf{R}$.

We'd like to assert that the functions $e^{r_1 x}$ and $e^{r_2 x}$ are solutions to the equation (1). If we assume that the law of exponents applies to complex numbers, then

$$e^{(\alpha+i\beta)x} = e^{\alpha x}e^{i\beta x}$$
$$e^{i\beta x} - ?$$

Let's assume that the Maclaurin series for e^z is the same for complex numbers z as it is for real numbers.

$$e^{i\theta} = \sum_{k=0}^{\infty} \frac{(i\theta)^k}{k!} = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \cdots + \frac{(i\theta)^k}{k!} + \cdots$$

Since $i^2 = -1$,

$$e^{i\theta} = 1 + (i\theta) - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} + \cdots =$$
$$\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right) = \cos\theta + i\sin\theta.$$

So,

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

then $e^{i\beta x} = \cos\beta x + i\sin\beta x$ and $e^{(\alpha+i\beta)x} = e^{\alpha x}e^{i\beta x} = e^{\alpha x}(\cos\beta x + i\sin\beta x)$.

Lemma 1. Let $z(x) = u(x) + iv(x)$ be a complex valued function of the real variable x, here $u(x)$ and $v(x)$ are real valued functions. And let $z(x)$ be a solution to the equation (1). Then, the functions $u(x)$ and $v(x)$ are real-valued solutions to the equation (1).

Proof. By assumption,

$$az'' + bz' + cz = a(u + iv)'' + b(u + iv)' + c(u + iv) =$$
$$a(u'' + iv'') + b(u' + iv') + c(u + iv) = (au'' + bu' + cu) + i(av'' + bv' + cv) = 0.$$

But a complex number $a + ib = 0$ if and only if $a = 0$ and $b = 0$. So,

$$au'' + bu' + cu = 0,$$
$$av'' + bv' + cv = 0,$$

which means that both $u(x)$ and $v(x)$ are real-valued solutions to (1).

When we apply Lemma 1 to the solution

$$\mathrm{e}^{(\alpha+i\beta)x} = \mathrm{e}^{\alpha x}(\cos \beta x + i \sin \beta x),$$

we obtain the following.

Complex conjugate roots.
If the auxiliary equation has complex conjugate roots $\alpha \pm i\beta$, then two linearly independent solutions to (1) are $\mathrm{e}^{\alpha x} \cos \beta x$ and $\mathrm{e}^{\alpha x} \sin \beta x$ and a general solution is

$$y(x) = c_1 \mathrm{e}^{\alpha x} \cos \beta x + c_2 \mathrm{e}^{\alpha x} \sin \beta x,$$

where c_1 and c_2 are arbitrary constants.

Example 2. Find a general solutions.

(a) $y'' - 10y' + 26y = 0$.

SOLUTION The associated auxiliary equation is

$$r^2 - 10r + 26 = 0,$$

which roots are $r_1 = -5+i$, $r_2 = -5-i$, the fundamental solution set is $\{\mathrm{e}^{-5x} \cos x, \mathrm{e}^{-5x} \sin x\}$. Thus, the general solution is

$$y(x) = \mathrm{e}^{-5x}(c_1 \cos x + c_2 \sin x).$$

(b) $y'' + 4y = 0$.

SOLUTION The associated auxiliary equation is

$$r^2 + 4 = 0,$$

which roots are $r_1 = 2i$, $r_2 = -2i$, the fundamental solution set is $\{\cos 2x, \sin 2x\}$. Thus, the general solution is

$$y(x) = c_1 \cos 2x + c_2 \sin 2x.$$

(c) $y''' - y'' + 4y' - 4y = 0$.

SOLUTION The associated auxiliary equation is

$$r^3 - r^2 + 4r - 4 = r^2(r-1) + 4(r-1) = (r^2+4)(r-1) = 0,$$

which has one real root $r_1 = 1$ and two complex roots $r_2 = 2i$, $r_3 = -2i$, so the fundamental solution set is $\{e^x, \cos 2x, \sin 2x\}$. Thus, the general solution is

$$y(x) = c_1 e^x + c_2 \cos 2x + c_3 \sin 2x.$$

Example 3. Solve the given initial value problems.

(a) $w'' - 4w' + 5w = 0$, $w(0) = 1$, $w'(0) = 4$.

SOLUTION The associated characteristic equation is

$$r^2 - 4r + 5 = 0,$$

which has two complex roots

$$r_1 = 2 + i, \quad r_2 = 2 - i.$$

So, the general solution to the given equation is

$$w(x) = e^{2x}(c_1 \cos x + c_2 \sin x).$$

To find the solution to the initial value problem we have to plug $x = 0$ into $w(x)$ and $w'(x)$.

$$w'(x) = 2e^{2x}(c_1 \cos x + c_2 \sin x) + e^{2x}(-c_1 \sin x + c_2 \cos x) = e^{2x}((2c_1 + c_2)\cos x + (2c_2 - c_1)\sin x),$$

$$w(0) = c_1 = 1,$$
$$w'(0) = 2c_1 + c_2 = 4.$$

Since $c_1 = 1$ and $c_2 = 4 - 2c_1 = 2$, the solution to the initial value problem is

$$w(x) = e^{2x}(\cos x + 2\sin x).$$

(b) $y'' + 9y = 0$, $y(0) = 1$, $y'(0) = 1$.
SOLUTION The associated characteristic equation is

$$r^2 + 9 = 0,$$

which has two complex roots

$$r_1 = 3i, \quad r_2 = -3i.$$

So, the general solution to the given equation is

$$y(x) = c_1 \cos 3x + c_2 \sin 3x.$$

To find the solution to the initial value problem we have to plug $x = 0$ into $y(x)$ and $y'(x)$.

$$y'(x) = -3c_1 \sin 3x + 3c_2 \cos 3x,$$
$$y(0) = c_1 = 1,$$
$$y'(0) = 3c_2 = 1.$$

Since $c_1 = 1$ and $c_2 = 1/3$, the solution to the initial value problem is

$$y(x) = \cos 3x + \frac{1}{3} \sin x.$$