Chapter 4. Linear Second Order Equations

Example 1. Find the general solution to the equation

$$y''' + y'' + 2y' - 4y = 0.$$

SOLUTION. The associated auxiliary equation is

$$r^3 + r^2 + 2r - 4 = 0,$$

which roots are $r_1 = 1$, $r_2 = -1 + i\sqrt{3}$, $r_3 = -1 - i\sqrt{3}$. So, the fundamental solution set to the equation is $\{e^x, e^{-x}\sin\sqrt{3}x, e^{-x}\cos\sqrt{3}x\}$. Thus, the general solution to the given equation is

$$y(x) = c_1 e^x + c_2 e^{-x}\sin\sqrt{3}x + c_3 e^{-x}\cos\sqrt{3}x.$$

Section 4.7 **Superposition and nonhomogeneous equations**

Let L be a linear differential operator

$$L[y](x) = y'' + p(x)y' + q(x)y.$$

L maps function y into the function $y'' + p(x)y' + q(x)y$.
Suppose that $y_1(x)$ an $y_2(x)$ are two functions, and

$$L[y_1](x) = y_1''(x) + p(x)y_1'(x) + q(x)y_1(x) = g_1(x),$$
$$L[y_2](x) = y_2''(x) + p(x)y_2'(x) + q(x)y_2(x) = g_2(x).$$

Since L is a linear operator,

$$L[c_1y_1 + c_2y_2](x) = c_1L[y_1](x) + c_2L[y_2](x) = c_1g_1(x) + c_2g_2(x).$$

Theorem 1 (superposition principle) Let y_1 be a solution to a differential equation

$$L[y_1](x) = y_1''(x) + p(x)y_1'(x) + q(x)y_1(x) = g_1(x)$$

and let y_2 be a solution to a differential equation

$$L[y_2](x) = y_2''(x) + p(x)y_2'(x) + q(x)y_2(x) = g_2(x),$$

where L is a linear differential operator. Then for any constant c_1 and c_2, the function $c_1y_1 + c_2y_2$ is the solution to the differential equation

$$L[y](x) = c_1g_1(x) + c_2g_2(x).$$

Example 2. Given that $y_1(x) = \cos x$ is a solution to

$$y'' - y' + y = \sin x,$$

and $y_2(x) = \frac{e^{2x}}{3}$ is a solution to

$$y'' - y' + y = e^{2x},$$

find solutions to the following differential equations:
(a) $y'' - y' + y = 5\sin x$;

SOLUTION. Let $g_1(x) = \sin x$, $g_2(x) = e^{2x}$. Since $5\sin x = 5g_1(x)$, the solution to the given equation is

$$y(x) = 5y_1(x) = 5\cos x.$$

(b) $y'' - y' + y = 4\sin x + 18e^{2x}$.

SOLUTION. Since $4\sin x + 18e^{2x} = 4g_1(x) + 18g_2(x)$, the solution to the given equation is

$$y(x) = 4y_1(x) + 18y_2(x) = 4\cos x + 18\frac{e^{2x}}{3} = 4\cos x + 6e^{2x}.$$

Theorem 2 (representation of solutions for nonhomogeneous equations). Let $y_p(x)$ be particular solution to the nonhomogeneous equation

$$y'' + p(x)y' + q(x)y = g(x) \tag{1}$$

on the interval (a, b) and let $y_1(x)$ and $y_2(x)$ be linearly independent solutions on (a, b) of the corresponding homogeneous equation

$$y'' + p(x)y' + q(x)y = 0.$$

Then a **general solution** of (1) on the interval (a, b) can be expressed in form

$$y(x) = y_p(x) + c_1y_1(x) + c_2y_2(x). \tag{2}$$

Procedure for solving solving nonhomogeneous equations.
To solve $y'' + p(x)y' + q(x)y = g(x)$:
(a) Determine the general solution $c_1y_1(x) + c_2y_2(x)$ of the corresponding homogeneous equation.
(b) Find the particular solution $y_p(x)$ of the given nonhomogeneous equation.
(c) Form the sum of the particular solution and a general solution to the homogeneous equation

$$y(x) = y_p(x) + c_1y_1(x) + c_2y_2(x),$$

to obtain the general solution to the given equation.

Example 3. A nonhomogeneous equation and a particular solution are given. Find a general solution to the equation.

(a) $y'' - 2y' + y = 8e^x$, $y_p(x) = 4x^2e^x$.

SOLUTION. The corresponding homogeneous equation

$$y'' - 2y' + y = 0,$$

has the auxiliary equation

$$r^2 - 2r + 1 = (r-1)^2 = 0.$$

Since the auxiliary equation has one repeated root $r = 1$, general solution to homogeneous equation is

$$y_h(x) = (c_1 + c_2 x)\mathrm{e}^x$$

.

Thus, the general solution to the nonhomogeneous equation is

$$y(x) = 4x^2\mathrm{e}^x + (c_1 + c_2 x)\mathrm{e}^x$$

.

(b) $y'' + 4y = \sin x - \cos x$, $y_p(x) = \frac{1}{3}(\sin x - \cos x)$.

SOLUTION. The corresponding homogeneous equation

$$y'' + 4y = 0,$$

has the auxiliary equation

$$r^2 + 4 = 0.$$

Since the auxiliary equation has complex conjugate roots $r_1 = 2i$, $r_2 = -2i$, general solution to homogeneous equation is

$$y_h(x) = c_1 \sin 2x + c_2 \cos 2x$$

.

Thus, the general solution to the nonhomogeneous equation is

$$y(x) = \frac{1}{3}(\sin x - \cos x) + c_1 \sin 2x + c_2 \cos 2x$$

.

(c) $y'' + y' - 12y = x^2 - 1$, $y_p(x) = -\frac{x^2}{2} - \frac{x}{2} + \frac{5}{4}$.

SOLUTION. The corresponding homogeneous equation

$$y'' + y' - 12y = 0,$$

has the auxiliary equation

$$r^2 + r - 12 = 0.$$

Since the auxiliary equation has two real roots $r_1 = 3$, $r_2 = -4$, general solution to homogeneous equation is

$$y_h(x) = c_1\mathrm{e}^{3x} + c_2\mathrm{e}^{-4x}$$

.

Thus, the general solution to the nonhomogeneous equation is

$$y(x) = -\frac{x^2}{2} - \frac{x}{2} + \frac{5}{4} + c_1 e^{3x} + c_2 e^{-4x}$$

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