Chapter 4. Linear Second Order Equations

Section 4.8 **Method of Undetermined Coefficients**

In this section, we give a simple procedure for finding a particular solution to the equation

$$ay'' + by' + cy = g(x), \qquad (1)$$

when the nonhomogeneous term $g(x)$ is of a special form

$$g(x) = e^{\alpha x}(P_{m_1}(x)\cos\beta x + Q_{m_2}(x)\sin\beta x),$$

where

$$P_{m_1}(x) = p_0x^{m_1} + p_1x^{m_1-1} + p_2x^{m_1-2} + \ldots + p_{m_1-1}x + p_{m_1}$$

is a polynomial of degree m_1 and

$$Q_{m_2}(x) = q_0x^{m_2} + q_1x^{m_2-1} + q_2x^{m_2-2} + \ldots + q_{m_2-1}x + q_{m_2}$$

is a polynomial of degree m_2, $\alpha, \beta \in \mathbf{R}$.

To apply the method of undetermined coefficients, we first have to solve the auxiliary equation for the corresponding homogeneous equation

$$ar^2 + br + c = 0$$

.

Let $\alpha = \beta = 0$, then

$$g(x) = p_0x^{m_1} + p_1x^{m_1-1} + p_2x^{m_1-2} + \ldots + p_{m_1-1}x + p_{m_1}.$$

We seek a particular solution of the form

$$y_p(x) = Ax^{m_1} + Bx^{m_1-1} + Cx^{m_1-2} + \ldots + Dx + F,$$

if $r = 0$ is **not** a root to the auxiliary equation. Here A, B, C, D, and F are unknown numbers.

If $r = 0$ is **one of two** roots of the auxiliary equation, then the particular solution is

$$y_p(x) = x(Ax^{m_1}+Bx^{m_1-1}+Cx^{m_1-2}+\ldots+Dx+F) = Ax^{m_1+1}+Bx^{m_1}+Cx^{m_1-1}+\ldots+Dx^2+Fx.$$

If $r = 0$ is a **repeated** root to the auxiliary equation, then the particular solution is

$$y_p(x) = x^2(Ax^{m_1}+Bx^{m_1-1}+Cx^{m_1-2}+\ldots+Dx+F) = Ax^{m_1+2}+Bx^{m_1+1}+Cx^{m_1}+\ldots+Dx^3+Fx^2.$$

To find unknowns A, B, C,..., D, and F, we have to substitute $y_p(x)$, $y_p'(x)$, and $y_p''(x)$ into equation (1). Set the corresponding coefficients from both sides of this equation to each other to form a system of linear equations with unknowns A, B, C,..., D, and F. Solve the system of linear equation for A, B, C,..., D, and F.

Example 1. Find the general solution to the equation

$$y'' + y' = 1 - 2x^2.$$

SOLUTION. Let's find the general solution to the corresponding homogeneous equation

$$y'' + y' = 0.$$

The associated auxiliary equation is

$$r^2 + r = r(r + 1) = 0,$$

which has two roots $r = 0$ and $r = -1$. Thus, the general solution to the homogeneous equation is

$$y_h(x) = c_1 + c_2 \mathrm{e}^{-x}.$$

Since $r = 0$ is one of two roots to the auxiliary equation and $m_1 = 2$, we seek a particular solution to the nonhomogeneous equation of the form

$$y_p(x) = x(Ax^2 + Bx + C) = Ax^3 + Bx^2 + Cx,$$

where A, B, and C are unknowns.

Now we have to substitute $y_p(x)$, $y_p'(x)$, and $y_p''(x)$ into equation.

$$y_p'(x) = 3Ax^2 + 2Bx + C,$$

$$y_p''(x) = 6Ax + 2B,$$

$$y_p''(x) + y_p'(x) = 6Ax + 2B + 3Ax^2 + 2Bx + C = 3Ax^2 + (6A + 2B)x + (2B + C) = 1 - 2x^2.$$

Two polynomials are equal when corresponding coefficients are equal, so we set

$$\begin{aligned} x^2 &: \quad 6A = -2, \\ x^1 &: \quad 6A + 2B = 0, \\ x^0 &: \quad 2B + C = 1 \end{aligned}$$

Solving the system gives $A = -1/3$, $B = -3A = 1$, $C = 1 - 2B = 1 - 2 = -1$. So,

$$y_p(x) = -\frac{1}{3}x^3 + x^2 - x,$$

and the general solution to the given nonhomogeneous equation is

$$y(x) = -\frac{1}{3}x^3 + x^2 - x + c_1 + c_2 \mathrm{e}^{-x}.$$

Let $\alpha \neq 0, \beta = 0$, then

$$g(x) = e^{\alpha x}(p_0 x^{m_1} + p_1 x^{m_1-1} + p_2 x^{m_1-2} + \ldots + p_{m_1-1}x + p_{m_1}).$$

We seek a particular solution of the form

$$y_p(x) = e^{\alpha x}(Ax^{m_1} + Bx^{m_1-1} + Cx^{m_1-2} + \ldots + Dx + F),$$

if $r = \alpha$ is **not** a root to the auxiliary equation. Here A, B, C, D, and F are unknown numbers.

If $r = \alpha$ is **one of two** roots of the auxiliary equation, then the particular solution is

$$y_p(x) = e^{\alpha x}x(Ax^{m_1} + Bx^{m_1-1} + Cx^{m_1-2} + \ldots + Dx + F)$$

$$= e^{\alpha x}(Ax^{m_1+1} + Bx^{m_1} + Cx^{m_1-1} + \ldots + Dx^2 + Fx).$$

If $r = \alpha$ is a **repeated** root to the auxiliary equation, then the particular solution is

$$y_p(x) = e^{\alpha x}x^2(Ax^{m_1} + Bx^{m_1-1} + Cx^{m_1-2} + \ldots + Dx + F)$$

$$= e^{\alpha x}(Ax^{m_1+2} + Bx^{m_1+1} + Cx^{m_1} + \ldots + Dx^3 + Fx^2).$$

To find unknowns A, B, C,..., D, and F, we have to substitute $y_p(x)$, $y_p'(x)$, and $y_p''(x)$ into equation (1), set the corresponding coefficients from both sides of this equation to each other to form a system of linear equations with unknowns A, B, C,..., D, and F and solve the system of linear equation for A, B, C,..., D, and F.

Example 2. Find the general solution to the equation

$$y'' - 2y' = 2e^{-2x}.$$

SOLUTION. Let's find the general solution to the corresponding homogeneous equation

$$y'' - 2y' = 0.$$

The associated auxiliary equation is

$$r^2 - 2r = r(r-2) = 0,$$

which has two roots $r = 0$ and $r = 2$. Thus, the general solution to the homogeneous equation is

$$y_h(x) = c_1 + c_2 e^{2x}.$$

Since $r = -2$ is not a root to the auxiliary equation and $m_1 = 0$, we seek a particular solution to the nonhomogeneous equation of the form

$$y_p(x) = Ae^{-2x}.$$

where A is unknown.

Now we have to substitute $y_p(x)$, $y_p'(x)$, and $y_p''(x)$ into equation.

$$y_p'(x) = -2Ae^{-2x},$$

$$y_p''(x) = 4A\mathrm{e}^{-2x},$$

$$y_p''(x) - 2y_p'(x) = 4A\mathrm{e}^{-2x} + 4A\mathrm{e}^{-2x} = 8A\mathrm{e}^{-2x} = 2\mathrm{e}^{-2x}.$$

Dividing by e^{-2x} gives $8A = 2$ or $A = 1/4$. So,

$$y_p(x) = \frac{1}{4}\mathrm{e}^{-2x},$$

and the general solution to the given nonhomogeneous equation is

$$y(x) = \frac{1}{4}\mathrm{e}^{-2x} + c_1 + c_2\mathrm{e}^{2x}.$$

Example 3. Find the general solution to the equation

$$y'' - 4y' + 4y = 16x^2\mathrm{e}^{2x}.$$

SOLUTION. Let's find the general solution to the corresponding homogeneous equation

$$y'' - 4y' + 4y = 0.$$

The associated auxiliary equation is

$$r^2 - 4r + 4 = (r-2)^2 = 0,$$

which has one repeated root $r = 2$. Thus, the general solution to the homogeneous equation is

$$y_h(x) = (c_1 + c_2x)\mathrm{e}^{2x}.$$

Since $r = 2$ is a repeated root to the auxiliary equation and $m_1 = 2$, we seek a particular solution to the nonhomogeneous equation of the form

$$y_p(x) = x^2(Ax^2 + Bx + C)\mathrm{e}^{2x} = (Ax^4 + Bx^3 + Cx^2)\mathrm{e}^{2x}.$$

where A, B, and C are unknowns.

Now we have to substitute $y_p(x)$, $y_p'(x)$, and $y_p''(x)$ into equation.

$$y_p'(x) = (2Ax^4 + (4A+2B)x^3 + (3B+2C)x^2 + 2Cx)\mathrm{e}^{2x},$$

$$y_p''(x) = (4Ax^4 + (16A+4B)x^3 + (12A+12B+4C)x^2 + (6B+8C)x + 2C)\mathrm{e}^{2x},$$

$$y_p''(x) - 4y_p'(x) + 4y_p =$$
$$= (4Ax^4 + (16A+4B)x^3 + (12A+12B+4C)x^2 + (6B+8C)x + 2C)\mathrm{e}^{2x} -$$
$$-4(2Ax^4 + (4A+2B)x^3 + (3B+2C)x^2 + 2Cx)\mathrm{e}^{2x} + 4(Ax^4 + Bx^3 + Cx^2)\mathrm{e}^{2x} = 16x^2\mathrm{e}^{2x}.$$

Dividing by e^{2x} gives

$$x^4(4A-8A+4A)+x^3(4B+16A-8B-16A+4B)+x^2(12A+12B+4C-8C-12B+4C)+$$

$$+x(8C+6B-8C)+2C=12Ax^2+6Bx+2C=16x^2.$$

Two polynomials are equal when corresponding coefficients are equal, so we set

$$\begin{aligned} x^2:&\quad 12A=16,\\ x^1:&\quad 6B=0,\\ x^0:&\quad 2C=0 \end{aligned}$$

Solving the system gives $A=4/3$, $B=0$, $C=0$. So,

$$y_p(x)=\frac{4}{3}x^4\mathrm{e}^{2x},$$

and the general solution to the given nonhomogeneous equation is

$$y(x)=\frac{4}{3}x^4\mathrm{e}^{2x}+(c_1+c_2x)\mathrm{e}^{2x}=\left(\frac{4}{3}x^4+c_1+c_2x\right)\mathrm{e}^{2x}.$$