Chapter 4. Linear Second Order Equations

Section 4.8 **Method of Undetermined Coefficients**

In this section, we give a simple procedure for finding a particular solution to the equation

$$ay'' + by' + cy = g(x), \qquad (1)$$

when the nonhomogeneous term $g(x)$ is of a special form

$$g(x) = \mathrm{e}^{\alpha x}(P_{m_1}(x)\cos\beta x + Q_{m_2}(x)\sin\beta x),$$

where

$$P_{m_1}(x) = p_0x^{m_1} + p_1x^{m_1-1} + p_2x^{m_1-2} + \ldots + p_{m_1-1}x + p_{m_1}$$

is a polynomial of degree m_1 and

$$Q_{m_2}(x) = q_0x^{m_2} + q_1x^{m_2-1} + q_2x^{m_2-2} + \ldots + q_{m_2-1}x + q_{m_2}$$

is a polynomial of degree m_2, $\alpha, \beta \in \mathbf{R}$.

To apply the method of undetermined coefficients, we first have to solve the auxiliary equation for the corresponding homogeneous equation

$$ar^2 + br + c = 0$$

.

Let $\alpha = 0, \beta \neq 0$, then

$$g(x) = (p_0x^{m_1} + p_1x^{m_1-1} + p_2x^{m_1-2} + \ldots + p_{m_1-1}x + p_{m_1})\cos\beta x+$$
$$+(q_0x^{m_2} + q_1x^{m_2-1} + q_2x^{m_2-2} + \ldots + q_{m_2-1}x + q_{m_2})\sin\beta x.$$

We seek a particular solution of the form

$$y_p(x) = (A_0x^m + A_1x^{m-1} + A_2x^{m-2} + \ldots + A_{m-1}x + A_m)\cos\beta x+$$
$$+(B_0x^m + B_1x^{m-1} + B_2x^{m-2} + \ldots + B_{m-1}x + B_m)\sin\beta x,$$

if $i\beta$ is **not** a root to the auxiliary equation. Here $m = \max\{m_1, m_2\}$, A_0,...,A_m, B_0,..., B_m are unknowns.

If $i\beta$ is **one of two** roots of the auxiliary equation, then the particular solution is

$$y_p(x) = x(A_0x^m + A_1x^{m-1} + A_2x^{m-2} + \ldots + A_{m-1}x + A_m)\cos\beta x+$$
$$+x(B_0x^m + B_1x^{m-1} + B_2x^{m-2} + \ldots + B_{m-1}x + B_m)\sin\beta x =$$
$$= (A_0x^{m+1} + A_1x^m + A_2x^{m-1} + \ldots + A_{m-1}x^2 + A_mx)\cos\beta x+$$
$$+(B_0x^{m+1} + B_1x^m + B_2x^{m-1} + \ldots + B_{m-1}x^2 + B_mx)\sin\beta x.$$

To find unknowns A_0,...,A_m, B_0,..., B_m, we have to substitute $y_p(x)$, $y_p'(x)$, and $y_p''(x)$ into equation (1). Set the corresponding coefficients from both sides of this equation to each other to form a system of linear equations with unknowns A_0,...,A_m, B_0,..., B_m. Solve the system of linear equation for A_0,...,A_m, B_0,..., B_m.

Example 1. Find the solution to the initial value problem

$$y'' - 3y' + 2y = 4x\sin x, \qquad y(0) = 3, \ \ y'(0) = 2.$$

SOLUTION. Let's find the general solution to the corresponding homogeneous equation

$$y'' - 3y' + 2y = 0.$$

The associated auxiliary equation is

$$r^2 - 3r + 2 = 0,$$

which has two roots $r_1 = 1$ and $r_2 = 2$. Thus, the general solution to the homogeneous equation is

$$y_h(x) = c_1 e^x + c_2 e^{2x}.$$

Since $r = i$ is not a root to the auxiliary equation and $m_1 = 1$, $m_2 = 0$, $m = 1$ we seek a particular solution to the nonhomogeneous equation of the form

$$y_p(x) = (Ax + B)\cos x + (Cx + D)\sin x,$$

where A, B, C, and D are unknowns.

Now we have to substitute $y_p(x)$, $y_p'(x)$, and $y_p''(x)$ into equation.

$$y_p'(x) = (Cx + A + D)\cos x + (C - Ax - B)\sin x,$$

$$y_p''(x) = (2C - Ax - B)\cos x + (-2A - Cx - D)\sin x,$$

$$y_p''(x) - 3y_p'(x) + 2y_p(x) = (2C - Ax - B)\cos x + (-2A - Cx - D)\sin x -$$
$$-3((Cx + A + D)\cos x + (C - Ax - B)\sin x) + 2((Ax + B)\cos x + (Cx + D)\sin x) =$$
$$= (4Ax - 4C + 4B)\cos x + (4Cx + 4D + 4A)\sin x = 4x\sin x.$$

We set

$$\begin{aligned} x\cos x: &\quad 4A = 0, \\ \cos x: &\quad -4C + 4B = 0, \\ x\sin x: &\quad 4C = 4, \\ \sin x: &\quad 4D + 4A = 0. \end{aligned}$$

Solving the system gives $A = 0$, $C = 1$, $B = C = 1$, $D = -A = 0$. So,

$$y_p(x) = \cos x + x\sin x$$

and the general solution to the given nonhomogeneous equation is

$$y(x) = \cos x + x\sin x + c_1 e^x + c_2 e^{2x}.$$

Substituting $y(x)$ into initial conditions gives

$$y(0) = 1 + c_1 + c_2 = 3,$$

$$y'(x) = -\sin x + \sin x - x\cos x + c_1 e^x + 2c_2 e^{2x} = -x\cos x + c_1 e^x + 2c_2 e^{2x},$$

$$y'(0) = c_1 + 2c_2 = 2.$$

Solving the system

$$\begin{cases} 1 + c_1 + c_2 = 3, \\ c_1 + 2c_2 = 2 \end{cases}$$

gives $c_1 = 2$, $c_2 = 0$. Thus, the solution to the initial value problem is

$$y(x) = \cos x + x\sin x + 2e^x.$$

Let $\alpha \neq 0, \beta \neq 0$, then

$$g(x) = e^{\alpha x}[(p_0 x^{m_1} + p_1 x^{m_1-1} + p_2 x^{m_1-2} + \ldots + p_{m_1-1}x + p_{m_1})\cos\beta x +$$
$$+ (q_0 x^{m_2} + q_1 x^{m_2-1} + q_2 x^{m_2-2} + \ldots + q_{m_2-1}x + q_{m_2})\sin\beta x].$$

We seek a particular solution of the form

$$y_p(x) = e^{\alpha x}[(A_0 x^m + A_1 x^{m-1} + A_2 x^{m-2} + \ldots + A_{m-1}x + A_m)\cos\beta x +$$
$$+ (B_0 x^m + B_1 x^{m-1} + B_2 x^{m-2} + \ldots + B_{m-1}x + B_m)\sin\beta x],$$

if $\alpha + i\beta$ is **not** a root to the auxiliary equation. Here $m = \max\{m_1, m_2\}$, A_0,...,A_m, B_0,..., B_m are unknowns.

If $\alpha + i\beta$ is **one of two** roots of the auxiliary equation, then the particular solution is

$$y_p(x) = e^{\alpha x}x[(A_0 x^m + A_1 x^{m-1} + A_2 x^{m-2} + \ldots + A_{m-1}x + A_m)\cos\beta x +$$
$$+ x(B_0 x^m + B_1 x^{m-1} + B_2 x^{m-2} + \ldots + B_{m-1}x + B_m)\sin\beta x] =$$
$$= e^{\alpha x}[(A_0 x^{m+1} + A_1 x^m + A_2 x^{m-1} + \ldots + A_{m-1}x^2 + A_m x)\cos\beta x +$$
$$+ (B_0 x^{m+1} + B_1 x^m + B_2 x^{m-1} + \ldots + B_{m-1}x^2 + B_m x)\sin\beta x].$$

To find unknowns A_0,...,A_m, B_0,..., B_m, we have to substitute $y_p(x)$, $y_p'(x)$, and $y_p''(x)$ into equation (1). Set the corresponding coefficients from both sides of this equation to each other to form a system of linear equations with unknowns A_0,...,A_m, B_0,..., B_m. Solve the system of linear equation for A_0,...,A_m, B_0,..., B_m.

Example 2. Find a general solution to the equation

$$y'' - 9y = e^{3x}\cos x.$$

SOLUTION. Let's find the general solution to the corresponding homogeneous equation

$$y'' - 9y = 0.$$

The associated auxiliary equation is

$$r^2 - 9 = 0,$$

which has two roots $r_1 = 3$ and $r_2 = -3$. Thus, the general solution to the homogeneous equation is

$$y_h(x) = c_1 \mathrm{e}^{3x} + c_2 \mathrm{e}^{-3x}.$$

Since $r = 3+i$ is not a root to the auxiliary equation and $m_1 = m_2 = 0$, we seek a particular solution to the nonhomogeneous equation of the form

$$y_p(x) = \mathrm{e}^{3x}(A\cos x + B\sin x),$$

where A and B are unknowns.
Now we have to substitute $y_p(x)$ and $y_p''(x)$ into equation.

$$y_p'(x) = \mathrm{e}^{3x}[(3A+B)\cos x + (3B-A)\sin x],$$

$$y_p''(x) = \mathrm{e}^{3x}[(8A+6B)\cos x + (8B-6A)\sin x],$$

$$y_p''(x) - 9y_p(x) = \mathrm{e}^{3x}[(8A+6B)\cos x + (8B-6A)\sin x] - 9\mathrm{e}^{3x}(A\cos x + B\sin x) =$$

$$\mathrm{e}^{3x}[(6B-A)\cos x + (-6A-B)\sin x] = \mathrm{e}^{3x}\cos x.$$

We set

$$\begin{array}{ll} \mathrm{e}^{3x}\cos x: & 6B - A = 1, \\ \mathrm{e}^{3x}\sin x: & -6A - B = 0. \end{array}$$

Solving the system gives $A = -1/37$, $B = -6A = 6/37$. So,

$$y_p(x) = \mathrm{e}^{3x}\left(-\frac{1}{37}\cos x + \frac{6}{37}\sin x\right),$$

and the general solution to the given nonhomogeneous equation is

$$y(x) = \mathrm{e}^{3x}\left(-\frac{1}{37}\cos x + \frac{6}{37}\sin x\right) + c_1\mathrm{e}^{3x} + c_2\mathrm{e}^{-3x}.$$

Particular solutions to $ay'' + by' + cy = g(x)$

Type	$g(x)$	$y_p(x)$
(I)	$p_0x^{m_1} + p_1x^{m_1-1} + \ldots + p_{m_1}$	$x^s(Ax^{m_1} + Bx^{m_1-1} + \ldots + Dx + F)$
(II)	$d\mathrm{e}^{\alpha x}$	$x^sA\mathrm{e}^{\alpha x}$
(III)	$\mathrm{e}^{\alpha x}(p_0x^{m_1} + p_1x^{m_1-1} + \ldots + p_{m_1})$	$x^s\mathrm{e}^{\alpha x}(Ax^{m_1} + Bx^{m_1-1} + \ldots + Dx + F)$
(IV)	$d\cos\beta x + f\sin\beta x$	$x^s(A\cos\beta x + B\sin\beta x)$
(V)	$(p_0x^{m_1} + p_1x^{m_1-1} + \ldots + p_{m_1})\cos\beta x +$ $+(q_0x^{m_2} + q_1x^{m_2-1} + \ldots + q_{m_2})\sin\beta x$	$x^s\{(A_0x^m + A_1x^{m-1} + \ldots + A_m)\cos\beta x +$ $+(B_0x^m + B_1x^{m-1} + \ldots + B_m)\sin\beta x\}$
(VI)	$\mathrm{e}^{\alpha x}(d\cos\beta x + f\sin\beta x)$	$x^s\mathrm{e}^{\alpha x}(A\cos\beta x + B\sin\beta x)$
(VII)	$\mathrm{e}^{\alpha x}[(p_0x^{m_1} + p_1x^{m_1-1} + \ldots + p_{m_1})\cos\beta x +$ $+(q_0x^{m_2} + q_1x^{m_2-1} + \ldots + q_{m_2})\sin\beta x]$	$x^s\mathrm{e}^{\alpha x}[(A_0x^m + A_1x^{m-1} + \ldots + A_m)\cos\beta x +$ $+(B_0x^m + B_1x^{m-1} + \ldots + B_m)\sin\beta x]$

In this table $s = 0$, when $\alpha + i\beta$ is not a root to the auxiliary equation, $s = 1$, when $\alpha + i\beta$ is one of two roots to the auxiliary equation, and $s = 0$, when $\beta = 0$ and α is a repeated root to the auxiliary equation; $m = \max\{m_1, m_2\}$.

Example 3. Using Table, find the form for a particular solution $y_p(x)$ to

$$y'' + y' - 2y = g(x),$$

where g(x) equals
(a) $g(x) = (2x^2 + 3)\mathrm{e}^{-2x}$,
(b) $g(x) = x\sin 2x$,
(c) $g(x) = \mathrm{e}^x + \cos 3x$.

SOLUTION. The auxiliary equation to the corresponding homogeneous equation is

$$r^2 + r - 2 = 0,$$

which has two roots $r_1 = -2$, $r_2 = 1$.

(a) From the table, function $g(x) = (2x^2 + 3)\mathrm{e}^{-2x}$ has type (III) with $\alpha = -2$ (one of two roots to the auxiliary equation), $m_1 = 2$. Hence y_p has the form

$$y_p(x) = x(Ax^2 + Bx + C)\mathrm{e}^{-2x} = (Ax^3 + Bx^2 + Cx)\mathrm{e}^{-2x}.$$

(b) From the table, function $g(x) = x\sin 2x$ has type (V) with $\beta = 2$, $m_1 = 0$, $m_2 = 1$. Hence y_p has the form

$$y_p(x) = (Ax + B)\cos 2x + (Bx + C)\sin 2x.$$

(c) In this case $y_p(x) = y_{1p}(x) + y_{2p}(x)$, where y_{1p} is a particular solution to the equation

$$y'' + y' - 2y = \mathrm{e}^x$$

and y_{2p} is a solution to

$$y'' + y' - 2y = \cos 3x.$$

Function $g_1(x) = \mathrm{e}^x$ has type (II) with $d = 1$, $\alpha = 1$ (one of two roots to the auxiliary equation), thus

$$y_{1p}(x) = Axe^x.$$

Function $g_2(x) = \cos 3x$ has type (IV) with $d = 1$, $f = 0$, $\beta = 3$, so

$$y_{2p}(x) = B\cos 3x + C\sin 3x.$$

The particular solution y_p has the form

$$y_p(x) = Axe^x + B\cos 3x + C\sin 3x.$$

Section 4.9 **Variation of Parameters**

Consider the nonhomogeneous linear second order differential equation

$$y'' + p(x)y' + q(x)y = g(x). \qquad (2)$$

Let $\{y_1(x), y_2(x)\}$ be a fundamental solution set to the corresponding homogeneous equation

$$y'' + p(x)y' + q(x)y = 0.$$

The general solution to this homogeneous equation is $y_h(x) = c_1y_1(x) + c_2y_2(x)$, where c_1 and c_2 are constants. To find a particular solution to (2) we assume that $c_1 = c_1(x)$ and $c_2 = c_2(x)$ are functions of x and we seek a particular solution $y_p(x)$ in form

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x).$$

Let's substitute $y_p(x)$, $y_p'(x)$, and $y_p''(x)$ into (2).

$$y_p'(x) = c_1'(x)y_1(x) + c_2'(x)y_2(x) + c_1(x)y_1'(x) + c_2(x)y_2'(x).$$

We simplify the computation assuming that

$$c_1'(x)y_1(x) + c_2'(x)y_2(x) = 0.$$

So,

$$y_p'(x) = c_1(x)y_1'(x) + c_2(x)y_2'(x).$$
$$y_p''(x) = c_1'(x)y_1'(x) + c_2'(x)y_2'(x) + c_1(x)y_1''(x) + c_2(x)y_2''(x).$$

$$y_p'' + p(x)y_p' + q(x)y_p = c_1'(x)y_1'(x) + c_2'(x)y_2'(x) + c_1(x)y_1''(x) + c_2(x)y_2''(x)+$$
$$+p(x)(c_1(x)y_1'(x) + c_2(x)y_2'(x)) + q(x)(c_1(x)y_1(x) + c_2(x)y_2(x)) =$$

$$= c_1'(x)y_1'(x) + c_2'(x)y_2'(x) + c_1(x)(y_1''(x) + p(x)y_1'(x) + q(x)y_1(x))+$$
$$+c_2(x)(y_2''(x) + p(x)y_2'(x) + q(x)y_2(x)) =$$

$$c_1'(x)y_1'(x) + c_2'(x)y_2'(x) = g(x).$$

To summarize, we can find $c_1(x)$ and $c_2(x)$ solving the system

$$\begin{cases} c_1'(x)y_1(x) + c_2'(x)y_2(x) = 0 \\ c_1'(x)y_1'(x) + c_2'(x)y_2'(x) = g(x) \end{cases}$$

for $c_1'(x)$ and $c_2'(x)$. Cramer's rule gives

$$c_1'(x) = \frac{-g(x)y_2(x)}{W[y_1, y_2](x)}, \qquad c_2'(x) = \frac{g(x)y_1(x)}{W[y_1, y_2](x)}.$$

Then

$$c_1(x) = \int \frac{-g(x)y_2(x)}{W[y_1, y_2](x)}dx, \qquad c_2(x) = \int \frac{g(x)y_1(x)}{W[y_1, y_2](x)}dx.$$

Example 4. Find the general solution to the equation

$$y'' + y = \frac{1}{\sin x}.$$

SOLUTION. The corresponding homogeneous equation is

$$y'' + y = 0.$$

The fundamental solution set to this homogeneous equation is $\{\cos x, \sin x\}$,

$$y_1(x) = \cos x, \qquad y_2(x) = \sin x,$$
$$y_1'(x) = -\sin x, \qquad y_2'(x) = \cos x,$$

The general solution to the homogeneous equation is

$$y_h(x) = c_1 \cos x + c_2 \sin x.$$

Then the particular solution to the nonhomogeheous equation is

$$y_p(x) = c_1(x) \cos x + c_2(x) \sin x.$$

To find $c_1(x)$ and $c_2(x)$ we have to solve the system

$$\begin{cases} c_1'(x) \cos x + c_2'(x) \sin x = 0 \\ c_1'(x)(-\sin x) + c_2'(x) \cos x = \frac{1}{\sin x}. \end{cases}$$

First equation gives $c_2' = -c_1' \frac{\cos x}{\sin x}$, so

$$c_1'(x)(-\sin x) + c_2'(x) \cos x = c_1'(x)(-\sin x) - c_1' \frac{\cos x}{\sin x} \cos x = -\frac{c_1'(\cos^2 x + \sin^2 x)}{\sin x} = -\frac{c_1'}{\sin x} = \frac{1}{\sin x},$$

$$c_1'(x) = -1,$$
$$c_1(x) = -x + c_3,$$

$$c_2' = -c_1' \frac{\cos x}{\sin x} = \frac{\cos x}{\sin x},$$
$$c_2(x) = \int \frac{\cos x}{\sin x}dx = \ln|\sin x| + c_4$$

.

Thus, the general solution to the given nonhomogneous equation is

$$y(x) = (-x + c_3)\cos x + (\ln|\sin x| + c_4)\sin x.$$