

Chapter 3. Mathematical methods and numerical methods involving first order equations.

Section 3.1 Mathematical modeling.

Formulate the problem

Here you must pose the problem in such a way that it can be "answered" mathematically.

Develop the model

There are two things to be one here. First, you must decide which variables are important and which are not. The former are then classified as independent variables or dependent variables. The unimportant variables are those that have very little or no effect on the process. The independent variables are those whose effect is significant. The dependent variables are those that are affected by the independent variables and that are important to solving the problem.

Second, you must determine or specify the relationships that exist among the relevant variables.

Test the model

The following questions should be answered:

Are the assumptions reasonable?

Are the equations dimensionally consistent?

Is the model internally consistent in the sense that equations do not contradict one another?

Do the relevant equation have solutions?

Are the solutions unique?

How difficult is to obtain the solutions?

Do the solutions provide an answer for the problem being studied?

You should always keep in mind that a model is not a reality but only representation of reality.

Section 3.2 Compartmental analysis

Many complicated processes can be broken down into distinct stages and the entire system modeled by describing the interactions between the various stages. Such systems are called **compartmental**.

The basic one-compartment system consists of a function $x(t)$ that represents the amount of a substance in the compartment at time t , an input rate at which the substance enters the compartment, and an output rate at which the substance leaves the compartment

$$\text{INPUT RATE} \longrightarrow x(t) \longrightarrow \text{OUTPUT RATE}$$

Because the derivative of x with respect to t can be interpreted as the rate of change in the amount of the substance in the compartment with respect to time, the one-compartment system suggests

$$\frac{dx}{dt} = \text{input rate} - \text{output rate}$$

as a mathematical model for the process.

Mixing problem.

Example 1. A brine solution of salt flows at a constant rate of $8L/min$ into a large tank that initially held $100L$ of brine solution in which was dissolved $0.5kg$ of salt. The solution inside the tank is kept well stirred and flows out of the tank in the same rate. If the concentration of salt in the brine entering the tank is $0.05kg/L$, determine the mass of salt in the tank after t min. When will the concentration of salt in the tank reach $0.02kg/L$?

SOLUTION. Let $x(t)$ denote the mass of salt in the tank at time t , we can determine the concentration of salt in the tank by dividing $x(t)$ by the volume of fluid in the tank at time t .

First, we must determine the rate at which salt enters the tank. We are given that brine flows into the tank at a rate of $8L/min$. Since the concentration is $0.05kg/L$, we conclude that the input rate of salt into the tank is

$$(8L/min)(0.05kg/L) = 0.4kg/min.$$

We must now determine the output rate of salt from the tank. The brine solution in the tank is well stirred, so let's assume that the concentration of salt in the tank is uniform. So, the concentration of salt in any part of the tank at time t is just $x(t)$ divided by the volume of fluid in the tank. Because the tank initially contains $100L$ and the rate of flow into the tank is the same as the rate of flow out, the volume is a constant $100L$. Hence, the output rate is

$$(8L/min) \left(\frac{x(t)}{100} kg/L \right) = \frac{2x(t)}{25} kg/min.$$

The tank initially contains $0.5kg$ of salt, so we set $x(0) = 0.5$. The initial value problem

$$\frac{dx}{dt} = \text{input rate} - \text{output rate} = 0.4 - \frac{2x(t)}{25}, \quad x(0) = 0.5,$$

is a mathematical model for the mixing problem.

Separating the variables and integrating gives

$$\begin{aligned} \frac{1}{2} \frac{dx}{5-x} &= \frac{1}{25} dt, \\ \frac{1}{2} \int \frac{dx}{5-x} &= \int \frac{1}{25} dt, \\ -\frac{1}{2} \ln |5-x| &= \frac{t}{25} + c, \\ 5-x &= c_1 e^{-\frac{2t}{25}}, \end{aligned}$$

$$x(t) = 5 - c_1 e^{-\frac{2t}{25}}.$$

Substituting into initial conditions gives

$$x(0) = 5 - c_1 = 0.5,$$

$$c_1 = 4.5$$

and the solution to the given initial problem is

$$x(t) = 5 - 4.5e^{-\frac{2t}{25}}.$$

To determine when the concentration of salt is 0.02 kg/L , we have to solve for t an equation

$$5 - 4.5e^{-\frac{2t}{25}} = 0.02,$$

$$e^{-\frac{2t}{25}} = \frac{8}{75},$$

$$t = -\frac{25}{2} \ln \frac{8}{75}.$$

Population models

Let $p(t)$ be the population of bacteria at time t . In our model we assume that the growth rate is proportional to the population present. We also assume that the death rate is zero. The mathematical model for population of bacteria is

$$\frac{dp}{dt} = k_1 p, \quad p(0) = p_0,$$

where $k_1 > 0$ is the proportionality constant for the growth rate and p_0 is the population at time $t = 0$.

For human population the assumption that the death rate is zero is wrong! If we assume that the people die only of natural causes, we might expect the death rate also to be proportional to the size of the population. So, we can rewrite formula

$$\frac{dp}{dt} = k_1 p - k_2 p = (k_1 - k_2)p = kp,$$

where $k = k_1 - k_2$ and k_2 is the proportionality constant for the death rate. Let's assume that $k_1 > k_2$ so that $k > 0$. This gives the mathematical model

$$\frac{dp}{dt} = kp, \quad p(0) = p_0,$$

which is called the **Malthusian** or **exponential, law** of population growth.

The solution to this initial value problem is

$$p(t) = p_0 e^{kt}.$$

Example 2. In 1980 the Department of Natural Resources released 1000 splake (a crossbreed of fish) into a lake. In 1987 the population of splake in the lake was estimated to be 3000. Using the Malthusian law, estimate the population of splake in the lake in the year 2010.

SOLUTION. In this case,

$$p_0 = 1000.$$

Then in 1987 the population of splake is

$$p(1987 - 1980) = 1000e^{7k} = 3000.$$

Solving this equation for k gives

$$k = \frac{\ln 3}{7},$$

$$\text{so } p(t) = 1000e^{\frac{t \ln 3}{7}} = 1000 \cdot 3^{\frac{t}{7}}.$$

In 2010 the population of splake will be

$$p(2010 - 1980) = p(40) = 1000 \cdot 3^{\frac{40}{7}}.$$

What about premature death? We might assume that another component of the death is proportional to the number of two-party interactions. There are $p(p-1)/2$ such possible interactions for a population of size p . Thus, if we combine the birth rate with the death rate and rearrange constants, we get the **logistic model**

$$\frac{dp}{dt} = -Ap(p - p_1), \quad p(0) = p_0,$$

where $A = k_3/2$ and $p_1 = (2k_1/k_3) + 1$.

This equation has two constant (equilibrium) solutions $p(t) = p_1$ and $p(t) = 0$. The nonequilibrium solutions can be found by separating variables

$$\int \frac{dp}{p(p - p_1)} = -A \int dt$$

or

$$\frac{1}{p_1} \ln \left| \frac{p - p_1}{p} \right| = -At + c$$

$$\left| 1 - \frac{p_1}{p} \right| = ce^{-Ap_1 t}.$$

If $p(0) = p_0$, and $c_3 = 1 - p_1/p_0$, then solving for $p(t)$, we find

$$p(t) = \frac{p_1}{1 - c_3 e^{-Ap_1 t}} = \frac{p_0 p_1}{p_0 + (p_1 - p_0) e^{-Ap_1 t}}.$$

The function $p(t)$ is called the **logistic function**.