Chapter 3. Mathematical methods and numerical methods involving first order equations.

Section 3.3 **Heating and cooling of buildings.**

Our goal is to formulate a mathematical model that describes the 24-hr temperature profile inside a building as a function of the outside temperature, the heat generated inside the building, and the furnace heating or air conditioner cooling. From this model we'd like to answer three following questions:

How long does it take to change the building temperature substantially?

How does the building temperature vary during spring and fall when there is no furnace heating or air conditioning?

How does the building temperature vary in summer when there is air conditioning or in the winter when there is furnace heating?

Let $T(t)$ represent the temperature inside the building at time t and view the building as a single compartment.

We will consider three main factors that affect the temperature inside the building. First is the heat produced by people, lights, and machines inside the building. This causes a rate of increase in temperature that we will denote by $H(t)$. Second is the heating (or cooling) supplied by the furnace (or air conditioner). This rate of increase (or decrease) in temperature will be represented by $U(t)$. The third factor is the effect of the outside temperature $M(t)$ on the temperature inside the building. Third factor can be modeled using **Newtons law of cooling**. This law states that

$$\frac{dT}{dt} = K(M(t) - T(t)).$$

The positive constant K depends on the physical properties of the building, K does not depend on M, T or t.

Summarizing, we have

$$\frac{dT}{dt} = K(M(t) - T(t)) + U(t) + H(t),$$

where $H(t) \geq 0$ and $U(t) > 0$ for furnace heating and $U(t) < 0$ for air conditioning cooling. Rewriting equation in the standard form

$$\frac{dT}{dt} + P(t)T(t) = Q(t),$$

where $P(t) = K$, $Q(t) = KM(t) + U(t) + H(t)$, integrating factor is

$$\mu(t) = \exp \int K dt = \mathrm{e}^{Kt},$$

then

$$\frac{d}{dt}[\mu(t)T(t)] = \mu(t)Q(t),$$

$$\frac{d}{dt}[\mathrm{e}^{Kt}T(t)] = \mathrm{e}^{Kt}Q(t),$$

$$T(t) = \mathrm{e}^{-Kt}\left\{\int \mathrm{e}^{Kt}Q(t)dt + C\right\} = \mathrm{e}^{-Kt}\left\{\int \mathrm{e}^{Kt}[KM(t) + U(t) + H(t)]dt + C\right\}.$$

Example 1. Suppose at the end of the day (at time t_0), when people leave the building, the outside temperature stays constant at M_0, the additional heating rate H inside of the building is zero, and the furnace/air conditioning rate U is also zero. Determine $T(t)$, given the initial condition $T(t_0) = T_0$.

SOLUTION. With $M = M_0$, $H = U = 0$, you can find $T(t)$ from the formula

$$T(t) = \mathrm{e}^{-Kt}\left\{\int \mathrm{e}^{Kt}KM_0dt + C\right\} = KM_0\mathrm{e}^{-Kt}\left\{\int \mathrm{e}^{Kt}dt + C\right\} = KM_0\mathrm{e}^{-Kt}\left(\frac{1}{K}\mathrm{e}^{Kt} + C\right) =$$

$$= M_0 + CKM_0\mathrm{e}^{-Kt} = M_0 + C_1\mathrm{e}^{-Kt}.$$

Setting $t = t_0$ gives

$$T(t_0) = M_0 + C_1\mathrm{e}^{-Kt_0} = T_0,$$

$$C_1 = (T_0 - M_0)\mathrm{e}^{Kt_0},$$

so the solution to this problem is

$$T(t) = M_0 + (T_0 - M_0)\mathrm{e}^{Kt_0}\mathrm{e}^{-Kt} = M_0 + (T_0 - M_0)\mathrm{e}^{-K(t-t_0)}.$$

When $M_0 < T_0$, the solution $T(t) = M_0 + (T_0 - M_0)\mathrm{e}^{-K(t-t_0)}$ decreases exponentially from the initial temperature T_0 to the final temperature M_0.

The constant $1/K$ is called **time constant of the building** (without heating or air conditioning). A typical value for the time constant of the building is 2 to 4 hr.

In the context of Example 1, we can use the notion of time constant to answer our initial question (a): The building temperature changes exponentially with a time constant of $1/K$. An answer to question (b) is given in the next example.

Example 2. Find the building temperature $T(t)$ if th additional heating rate $H(t) = H_0$, where H_0 is a constant, there is no hating or cooling, and the outside temperature varies as a sine wive over a 24-hr period, with its minimum at $t = 0$ (midnight) and its maximum at $t = 12$ (noon). That is,

$$M(t) = M_0 - B\cos\omega t,$$

where B is a positive constant, M_0 is a average outside temperature, and $\omega = 2\pi/24 = \pi/12$ radians/hr.

SOLUTION. The function $Q(t)$ is now

$$Q(t) = K(M_0 - B\cos\omega t) + H_0.$$

Setting $B_0 = M_0 + H_0/K$, we can rewrite Q as

$$Q(t) = K(B_0 - B\cos\omega t),$$

where KB_0 represents the daily average value of $Q(t)$; that is

$$KB_0 = \frac{1}{24}\int\limits_0^{24} Q(t)dt.$$

Then

$$T(t) = \mathrm{e}^{-Kt}\left\{\int \mathrm{e}^{Kt}K(B_0 - B\cos\omega t)dt + C\right\} = B_0 + C\mathrm{e}^{-Kt} - KB\mathrm{e}^{-Kt}\int \mathrm{e}^{Kt}\cos\omega t dt$$

Since

$$\int \mathrm{e}^{Kt}\cos\omega t dt = \frac{K\cos\omega t + \omega\sin\omega t}{K^2+\omega^2}\mathrm{e}^{Kt} = \frac{\cos\omega t + (\omega/K)\sin\omega t}{K(1+(\omega/K)^2)}\mathrm{e}^{Kt},$$

$$T(t) = B_0 + C\mathrm{e}^{-Kt} - B\frac{\cos\omega t + (\omega/K)\sin\omega t}{1+(\omega/K)^2} = B_0 + C\mathrm{e}^{-Kt} - BF(t),$$

where $F(t) = \frac{\cos\omega t + (\omega/K)\sin\omega t}{1+(\omega/K)^2}$.

The constant C is chosen so that at midnight ($t = 0$), the value of the temperature T is equal to some initial temperature T_0. Thus,

$$C = T_0 - B_0 + BF(0) = T_0 - B_0 + \frac{B}{1+(\omega/K)^2}.$$

Notice, that the term $C\mathrm{e}^{-Kt}$ tents to zero exponentially. We may assume that exponential term $C\mathrm{e}^{-Kt}$ has died out.

Typical value for the dimensionless ratio ω/K lie between 1/2 and 1. For this range, the lag between inside and outside temperature is approximately 1.8 to 3 hr and the magnitude of the inside variation is between 89% and 71% of the variation outside.

Example 3. Suppose, in the building in Example 2, a simple thermostat is installed that is used to compare the actual temperature inside the building with a desired temperature T_D. If the actual temperature is below the desired temperature, the furnace supplies heating; otherwise it is turned off. If the actual temperature is above the desired temperature, the furnace supplies cooling; otherwise it is off. Assuming that the amount of heating or cooling supplies is proportional to the difference in temperature–that is,

$$U(t) = K_U[T_D - T(t)],$$

where K_U is a positive proportionally constant. Find $T(t)$.

SOLUTION. After substituting $U(t)$, $M(t)$, and $H(t)$ into the differential equation for the building temperature, we get

$$\frac{dT}{dt} = K(M_0 - B\cos\omega t - T(t)) + K_U[T_D - T(t)] + H_0 =$$

$$= -(K + K_U)T(t) - K\cos\omega t + KM_0 + H_0 + K_U T_D,$$

So

$$Q(t) = -K\cos\omega t + KM_0 + H_0 + K_U T_D = K_1(B_2 - B_1\cos\omega t),$$

where $K_1 = K + K_U$, $B_2 = \frac{K_U T_D + KM_0 + H_0}{K_1}$, $B_1 = \frac{BK}{K_1}$.

Then

$$T(t) = \mathrm{e}^{-K_1 t}\left\{\int \mathrm{e}^{K_1 t} Q(t) dt + C\right\} = \mathrm{e}^{-K_1 t}\left\{\int \mathrm{e}^{K_1 t} K_1 (B_2 - B_1 \cos \omega t) dt + C\right\} =$$

$$= B_2 - B_1 F_1(t) + C\mathrm{e}^{-K_1 t},$$

where

$$F_1(t) = \frac{\cos \omega t + (\omega / K_1) \sin \omega t}{1 + (\omega / K_1)^2}.$$

The constant C is chosen so that at time $t = 0$, the value of the temperature T is equal to T_0. Thus,

$$C = T_0 - B_2 + B_1 F(0) = T_0 - B_2 + \frac{B_1}{1 + (\omega / K_1)^2}.$$

The constant $1/K_1$, where $K_1 = K + K_U$, is a **time constant for the building with heating and air conditioning**. For a typical heating an cooling system, K_U is somehow less than 2; for a typical building, constant K is between 1/2 an 1/4. Hence, the sum gives a value for K_1 of about 2, and the time constant for the building with heating and air conditioning is about 1/2.

When the heating or cooling is turned on, it takes about 30 minutes for the exponential term $C\mathrm{e}^{-K_1 t}$ to die off. If we neglect the exponential term, the average temperature inside the building is B_2. Since K_1 is much larger than K and H_0 is small, B_2 is roughly T_D. In other words, after certain period of time, the temperature inside of the building is roughly T_D with a small sinusoidal variation. Thus, to save energy, the heating or cooling system may be left off during the night. When it is turned on in the morning, it will take roughly 30 min for the inside of the building to attain the desired temperature.