Chapter I **Introduction**

Section 1.2 **Solutions and Initial Value Problems**

A general form for an nth-order equation with variable x and unknown function $y = y(x)$ can be expressed as

$$F\left(x, y, \frac{dy}{dx}, \cdots, \frac{d^n y}{dx^n}\right) = 0, \tag{1}$$

where F is a function that depends on x, y, and the derivatives of y up to the order n. We assume that the equation holds for all x in an open interval I ($a < x < b$, where a or b could be infinite). In many cases we can isolate the highest-order term $d^n y/dx^n$ and write equation (1) as

$$\frac{d^n y}{dx^n} = f\left(x, y, \frac{dy}{dx}, \cdots, \frac{d^{n-1} y}{dx^{n-1}}\right) = 0. \tag{2}$$

Definition 1. A function $\varphi(x)$ that when substituted by y in equation (1) or (2) satisfies the equation for all x in the interval I is called an *explicit solution* to the equation on I.

Examples Determine whether the function is an explicit solution to the given differential equation.

(a) $\varphi(x) = 5x^2$, $xy' = 2y$;

SOLUTION Since $\varphi(x) = 5x^2$, then $\varphi'(x) = 10x$. Substitution of $\varphi(x)$ for y in equation gives

$$x \cdot 10x = 2 \cdot 5x^2, \qquad \text{or}$$
$$10x^2 = 10x^2.$$

Since this is valid for any real x, the function $\varphi(x) = 5x^2$ is an explicit solution to equation on $(-\infty, +\infty)$.

(b) $\varphi(x) = C_1 \cos 5x + C_2 \sin 5x$, $y'' + 25y = 0$.

SOLUTION Lets compute

$$\varphi'(x) = \frac{d}{dx}(C_1 \cos 5x + C_2 \sin 5x) = -5C_1 \sin 5x + 5C_2 \cos 5x$$

and

$$\varphi''(x) = \frac{d}{dx}(-5C_1 \sin 5x + 5C_2 \cos 5x) = -25C_1 \cos 5x - 25C_2 \sin 5x.$$

Substitution of φ and φ'' for y and y'' in given equation yields

$$-25C_1 \cos 5x - 25C_2 \sin 5x + 25(C_1 \cos 5x + C_2 \sin 5x) = 0$$

for any C_1 and C_2.

(c) $\varphi(x) = \sqrt{2x - x^2}$, $y^3 y'' + 1 = 0$.

SOLUTION

$$\varphi'(x) = \frac{d}{dx}(\sqrt{2x - x^2}) = \frac{1}{2\sqrt{2x - x^2}}(2 - 2x) = \frac{1 - x}{\sqrt{2x - x^2}},$$

$$\varphi''(x) = \frac{d}{dx}\left(\frac{1 - x}{\sqrt{2x - x^2}}\right) = \frac{-\sqrt{2x - x^2} - (1 - x)\frac{1-x}{\sqrt{2x-x^2}}}{2x - x^2} =$$

$$\frac{-(2x - x^2) - (1 - x)^2}{(2x - x^2)^{3/2}} = -\frac{1}{(2x - x^2)^{3/2}}.$$

Then

$$\varphi^3(x)\varphi''(x) + 1 = (\sqrt{2x - x^2})^3 \cdot \left(-\frac{1}{(2x - x^2)^{3/2}}\right) + 1 = -1 + 1 = 0.$$

Thus, $\varphi(x) = \sqrt{2x - x^2}$ is an explicit solution for the given equation for all $0 < x < 2$.

Sometimes the solution to the equation is defined implicitly.

Example Show that the relation $x^2 - xy + y^2 = 1$ implicitly defines the solution to the equation $(x - 2y)y' = 2x - y$.

Lets find $y' = \frac{dy}{dx}$:

$$\frac{d}{dx}(x^2 - xy + y^2) = \frac{d}{dx}(1),$$

$$2x - y - x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0,$$

$$2x - y - (x - 2y)\frac{dy}{dx} = 0,$$

$$\frac{dy}{dx} = \frac{2x - y}{x - 2y}.$$

And now lets substitute the expression for y' into equation:

$$(x - 2y)\frac{2x - y}{x - 2y} = 2x - y,$$

$$2x - y = 2x - y$$

which is indeed valid for all $-\frac{2}{\sqrt{3}} < x < \frac{2}{\sqrt{3}}$.

Definition 2. A relation $G(x, y) = 0$ is said to be an *implicit solution* to equation (1) on the interval I if it defines one or more explicit solutions on I.

Example
Show that $y - \ln y = x^2 + 1$ is an implicit solution to the equation $\frac{dy}{dx} = \frac{2xy}{y-1}$.
SOLUTION Find y':

$$\frac{d}{dx}(y - \ln y) = \frac{d}{dx}(x^2 + 1),$$

$$y' - \frac{y'}{y} = 2x, \qquad y' = \frac{2xy}{y - 1},$$

which is identical to the given equation.

Example

Verify that $x^2 + Cy^2 = 1$, where C is an arbitrary constant, is a one-parameter family of implicit solution to

$$\frac{dy}{dx} = \frac{xy}{x^2 - 1}$$

and graph several of the solution curves using the same coordinate axes.

SOLUTION When we implicitly differentiate the equation $x^2 + Cy^2 = 1$ with respect to x, we find

$$2x + 2Cy\frac{dy}{dx} = 1, \qquad \text{or}$$

$$\frac{dy}{dx} = -\frac{x}{Cy}. \tag{3}$$

Since $x^2 - 1 = -Cy^2$, we can rewrite the given equation as follows:

$$\frac{dy}{dx} = \frac{xy}{x^2 - 1} = -\frac{xy}{Cy^2} = -\frac{x}{Cy},$$

which is identical to (3). In Figure 1 we have sketched the implicit solutions for $C = 0, \pm 1, \pm 2$. For $C = 0$, the implicit solution gives rise to two implicit solutions $x = 1$ and $x = -1$, for $C > 0$ the implicit solutions are ellipses with the center in (0,0), for $C < 0$, the curves are hyperbolas. Notice that the implicit solution curves fill the entire plane.

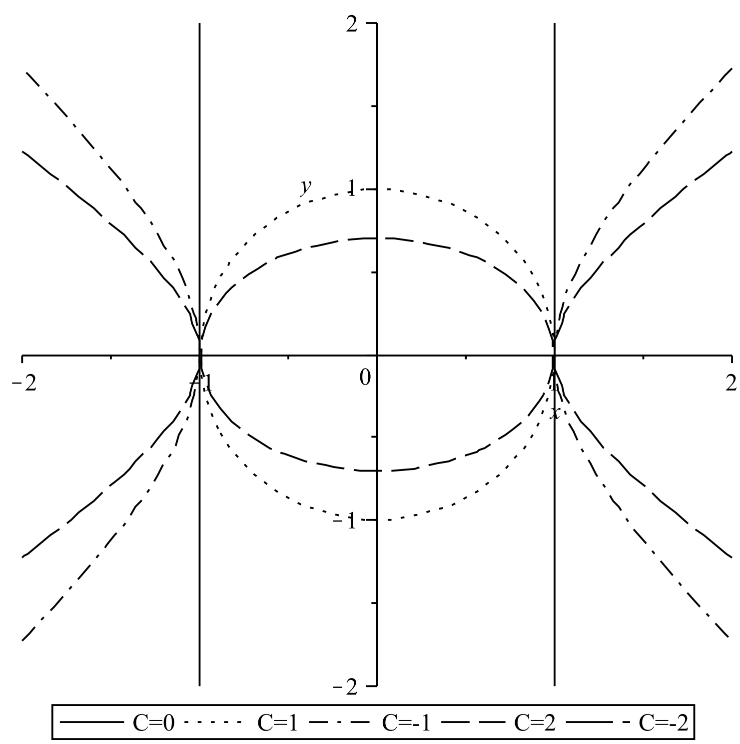

Figure 1 Implicit solutions $x^2 + Cy^2 = 1$