Chapter 7. Laplace Transforms.

Section 7.2 **Definition of the Laplace Transform.**

Definition 1. Let $f(x)$ be a function on $[0, \infty)$. The **Laplace transform** of f is the function F defined by the integral

$$F(s) = \int_0^\infty f(t)\mathrm{e}^{-st}dt.$$

The domain of $F(s)$ is all the values of s for which integral exists. The Laplace transform of f is denoted by both F and $\mathcal{L}\{f\}$.

Notice, that integral in definition is **improper** integral.

$$\int_0^\infty f(t)\mathrm{e}^{-st}dt = \lim_{N\to\infty} \int_0^N f(t)\mathrm{e}^{-st}dt$$

whenever the limit exists.

Example 1. Determine the Laplace transform of the given function.

(a) $f(t) = 1$, $t \geq 0$.

SOLUTION. Using the definition of Laplace transform, we compute

$$\mathcal{L}\{1\}(s) = \int_0^\infty 1 \cdot \mathrm{e}^{-st}dt = \lim_{N\to\infty} \int_0^N \mathrm{e}^{-st}dt = -\frac{1}{s} \lim_{N\to\infty} \mathrm{e}^{-st}\Big|_0^N = \frac{1}{s}.$$

So,

$$\mathcal{L}\{1\}(s) = \frac{1}{s}, \quad s > 0.$$

(b) $f(t) = t^2$, $t \geq 0$.

SOLUTION. Using the definition of Laplace transform, we compute

$$\mathcal{L}\{t^2\}(s) = \int_0^\infty t^2\mathrm{e}^{-st}dt = \lim_{N\to\infty} \int_0^N t^2\mathrm{e}^{-st}dt = \left| \begin{matrix} u = t^2 & u' = 2t \\ v' = \mathrm{e}^{-st} & v = -\frac{1}{s}\mathrm{e}^{-st} \end{matrix} \right| =$$

$$= \lim_{N\to\infty} \left[-\frac{t^2}{s}\mathrm{e}^{-st}\Big|_0^N + \frac{2}{s}\int_0^N t\mathrm{e}^{-st}dt \right] = \lim_{N\to\infty} \left[-\frac{N^2}{s}\mathrm{e}^{-sN} + \frac{2}{s}\int_0^N t\mathrm{e}^{-st}dt \right] =$$

$$= -\lim_{N\to\infty} \frac{N^2}{s}\mathrm{e}^{-sN} + \lim_{N\to\infty} \frac{2}{s}\int_0^N t\mathrm{e}^{-st}dt = \frac{2}{s} \lim_{N\to\infty} \int_0^N t\mathrm{e}^{-st}dt = \left| \begin{matrix} u = t & u' = 1 \\ v' = \mathrm{e}^{-st} & v = -\frac{1}{s}\mathrm{e}^{-st} \end{matrix} \right| =$$

$$= \frac{2}{s} \lim_{N\to\infty} \left[-\frac{t}{s}\mathrm{e}^{-st}\Big|_0^N + \frac{1}{s}\int_0^N \mathrm{e}^{-st}dt \right] = \frac{2}{s} \lim_{N\to\infty} \left[-\frac{N}{s}\mathrm{e}^{-sN} + \frac{1}{s}\int_0^N \mathrm{e}^{-st}dt \right] =$$

$$= -\frac{2}{s^2} \lim_{N\to\infty} N\mathrm{e}^{-sN} + \frac{2}{s^2} \lim_{N\to\infty} \int_0^N \mathrm{e}^{-st}dt = \frac{2}{s^2} \lim_{N\to\infty} \int_0^N \mathrm{e}^{-st}dt =$$

$$= -\frac{2}{s^3} \lim_{N\to\infty} \mathrm{e}^{-st}\Big|_0^N = \frac{2}{s^3}.$$

So,

$$\mathcal{L}\{t^2\}(s) = \frac{2}{s^3}, \quad s > 0.$$

(c) $f(t) = \mathrm{e}^{at}$, where a is a constant.

SOLUTION. Using the definition of Laplace transform, we compute

$$\mathcal{L}\{\mathrm{e}^{at}\}(s) = \int_0^\infty \mathrm{e}^{at}\mathrm{e}^{-st}dt = \lim_{N\to\infty} \int_0^N \mathrm{e}^{-(s-a)t}dt = -\frac{1}{s-a} \lim_{N\to\infty} \mathrm{e}^{-(s-a)t}\Big|_0^N = \frac{1}{s-a}.$$

So,

$$\mathcal{L}\{\mathrm{e}^{at}\}(s) = \frac{1}{s-a}, \quad s > a.$$

(d) $f(t) = \begin{cases} t^2, & 0 < t < 1, \\ 1, & 1 \le t \le 2, \\ 1-t, & 2 < t. \end{cases}$

SOLUTION. Since $f(t)$ is defined by a different formula on different intervals, we begin by breaking up the integral into three separate integrals.

$$\mathcal{L}\{f(t)\}(s) = \int_0^\infty f(t)\mathrm{e}^{-st}dt = \int_0^1 t^2\mathrm{e}^{-st}dt + \int_1^2 \mathrm{e}^{-st}dt + \int_2^\infty (1-t)\mathrm{e}^{-st}dt =$$

Since

$$\int_0^1 t^2\mathrm{e}^{-st}dt = \left(\frac{2}{s^3} - \frac{2}{s^2} - \frac{1}{s}\right)\mathrm{e}^{-s} - \frac{2}{s^3},$$

$$\int_1^2 \mathrm{e}^{-st}dt = \frac{1}{s}(\mathrm{e}^{-s} - \mathrm{e}^{-2s}),$$

$$\int_2^\infty (1-t)\mathrm{e}^{-st}dt = \left(\frac{1}{s} - \frac{1}{s^2}\right)\mathrm{e}^{-2s},$$

$$\mathcal{L}\{f(t)\}(s) = \left(\frac{2}{s^3} - \frac{2}{s^2} - \frac{1}{s}\right)\mathrm{e}^{-s} - \frac{2}{s^3} + \frac{1}{s}(\mathrm{e}^{-s} - \mathrm{e}^{-2s}) + \left(\frac{1}{s} - \frac{1}{s^2}\right)\mathrm{e}^{-2s} =$$

$$= \left(\frac{2}{s^3} - \frac{2}{s^2}\right) e^{-s} - \frac{2}{s^3} - \frac{1}{s^2} e^{-2s}.$$

The important property of the Laplace transform is its **linearity**. That is, the Laplace transform $\mathcal{L}$ is a linear operator.

Theorem 1. (linearity of the transform) Let f_1 and f_2 be functions whose Laplace transform exist for $s > \alpha$ and c_1 and c_2 be constants. Then, for $s > \alpha$,

$$\mathcal{L}\{c_1 f_1 + c_2 f_2\} = c_1 \mathcal{L}\{f_1\} + c_2 \mathcal{L}\{f_2\}.$$

Example 2. Determine $\mathcal{L}\{10 + 5e^{2t} + 3\cos 2t\}$.

SOLUTION.

$$\mathcal{L}\{10 + 5e^{2t} + 3\cos 2t\} = 10\mathcal{L}\{1\} + 5\mathcal{L}\{e^{2t}\} + 3\mathcal{L}\{\cos 2t\} = \frac{10}{s} + \frac{5}{s-5} + 3\mathcal{L}\{\cos 2t\}$$

Since

$$\mathcal{L}\{\cos bt\} = \int_0^\infty \cos bt e^{-st} dt = \lim_{N\to\infty} \int_0^N \cos bt e^{-st} dt = \frac{s}{s^2 + b^2},$$

$$\mathcal{L}\{10 + 5e^{2t} + 3\cos 2t\} = \frac{10}{s} + \frac{5}{s-5} + \frac{3s}{s^2+4}.$$

Existence of the transform.

There are functions for which the improper integral in Definition 1 fails to converge for any value of s. For example, no Laplace transform exists for the function e^{t^2}. Fortunately, the set of the functions for which the Laplace transform is defined includes many of the functions.

Definition 2. A function f is said to be **piecewise continuous on a finite interval** $[a, b]$ if f is continuous at every point in $[a, b]$, except possibly for a finite number of points at which $f(t)$ has a jump discontinuity.

A function $f(x)$ is said to be **piecewise continuous on** $[0, \infty)$ if $f(t)$ is piecewise continuous on $[0, N]$ for all $N > 0$.

Example 3. Show that function

$$f(t) = \begin{cases} \sin t, & 0 \le t \le \frac{\pi}{2}, \\ \frac{\pi+2-x}{2}, & \frac{\pi}{2} < t \le \pi + 2, \\ 3, & t > \pi + 2 \end{cases}$$

is piecewise continuous on $[0, \infty)$.

SOLUTION. $f(t)$ is continuous on the intervals $\left(0, \frac{\pi}{2}\right)$, $\left(\frac{\pi}{2}, \pi + 2\right)$, $(\pi + 2, \infty)$. The possible points of discontinuity are $t = \frac{\pi}{2}$ and $t = \pi + 2$. Let's find

$$\lim_{t\to\frac{\pi}{2}-0} f(t) = \lim_{t\to\frac{\pi}{2}} \sin t = 1,$$

$$\lim_{t\to\frac{\pi}{2}+0} f(t) = \lim_{t\to\frac{\pi}{2}} \frac{\pi + 2 - x}{2} = \frac{\pi}{4} + 1,$$

$$\lim_{t\to(\pi+2)-0} f(t) = \lim_{t\to(\pi+2)} \frac{\pi+2-x}{2} = 0,$$
$$\lim_{t\to(\pi+2)+0} f(t) = \lim_{t\to(\pi+2)} 3 = 3.$$

Since,

$$\lim_{t\to\frac{\pi}{2}-0} f(t) = 1 \neq \lim_{t\to\frac{\pi}{2}+0} f(t) = \frac{\pi}{4} + 1$$

and

$$\lim_{t\to(\pi+2)-0} f(t) = 0 \neq \lim_{t\to(\pi+2)+0} f(t) = 3,$$

$f(t)$ has jump discontinuities at $t = \frac{\pi}{2}$ and $t = \pi + 2$. Thus, f is piecewise continuous.

A function that is piecewise continuous on a *finite* interval is integrable over that interval. However, piecewise continuity on $[0, \infty)$ is not enough to guarantee the existence of the improper integral over $[0, \infty)$; we also need to consider the growth of the integrand as $t \to \infty$.

Definition 3. A function $f(t)$ is said to be of **exponential order** α if there exist positive constants T and M s.t.

$$|f(t)| \leq M\mathrm{e}^{\alpha t}, \text{ for all } t \geq T.$$

Theorem 2. If $f(t)$ is piecewise continuous on $t \to \infty$ and of exponential order α, then $\mathcal{L}\{f\}(s)$ exists for $s > \alpha$.

Brief table of Laplace transform

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$
1	$\frac{1}{s}, \quad s > 0$
e^{at}	$\frac{1}{s-a}, \quad s > a$
$t^n, \quad n = 1, 2, ...$	$\frac{n!}{s^{n+1}}, \quad s > 0$
$\sin bt$	$\frac{b}{s^2+b^2}, \quad s > 0$
$\cos bt$	$\frac{s}{s^2+b^2}, \quad s > 0$
$\mathrm{e}^{at}t^n, \quad n = 1, 2, ...$	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
$\mathrm{e}^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
$\mathrm{e}^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$