Chapter 7. Laplace Transforms.

Let $f(x)$ be a function on $[0, \infty)$. The **Laplace transform** of f is the function F defined by the integral

$$\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty f(t)\mathrm{e}^{-st}dt.$$

Brief table of Laplace transform

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$
1	$\frac{1}{s}, \quad s > 0$
e^{at}	$\frac{1}{s-a}, \quad s > a$
$t^n, \quad n = 1, 2, ...$	$\frac{n!}{s^{n+1}}, \quad s > 0$
$\sin bt$	$\frac{b}{s^2+b^2}, \quad s > 0$
$\cos bt$	$\frac{s}{s^2+b^2}, \quad s > 0$
$\mathrm{e}^{at}t^n, \quad n = 1, 2, ...$	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
$\mathrm{e}^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
$\mathrm{e}^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$

Section 7.3 **Properties of the Laplace Transform.**

Theorem 1. (translation in s) If the Laplace transform $\mathcal{L}\{f\}(s) = F(s)$ exist for $s > \alpha$, then

$$\mathcal{L}\{\mathrm{e}^{at}f\}(s) = F(s-a)$$

for $s > a + \alpha$.

Example 1. Determine

(a) $\mathcal{L}\{\mathrm{e}^{at}\cos bt\}$,

(b) $\mathcal{L}\{\mathrm{e}^{3t}t^2\}$,

(c) $\mathcal{L}\{\mathrm{e}^{2t}\sin^2 t\}$.

Theorem 2. (Laplace transform of the derivative) Let $f(t)$ be continuous on $[0, \infty)$ and $f'(t)$ be piecewise continuous on $[0, \infty)$, with both of exponential order α. Then, for $s > \alpha$

$$\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0).$$

Using induction, we can extend the last theorem to higher-order derivatives of $f(t)$.

$$\mathcal{L}\{f''\}(s) = s\mathcal{L}\{f'\}(s) - f'(0) = s(s\mathcal{L}\{f\}(s) - f(0)) - f'(0) = s^2\mathcal{L}\{f\}(s) - sf(0) - f'(0).$$

In general, we obtain the following result.

Theorem 3. (Laplace transform of higher-order derivatives) Let $f(t)$, $f'(t)$,...,$f^{(n-1)}(t)$ be continuous on $[0,\infty)$ and let $f^{(n)}(t)$ be be piecewise continuous on $[0,\infty)$, with all these function of exponential order α. Then, for $s > \alpha$

$$\mathcal{L}\{f^{(n)}\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0).$$

Example 2. Let $f(t)$ is continuous function. Prove that

$$\mathcal{L}\left\{\int\limits_0^t f(\tau)d\tau\right\}(s) = \frac{1}{s}\mathcal{L}\{f(t)\}(s).$$

If $F(s) = \mathcal{L}\{f\}(s)$ then

$$F'(s) = \mathcal{L}\{-tf(t)\}(s).$$

Theorem 4. (derivatives of the Laplace transform) Let $F(s) = \mathcal{L}\{f\}(s)$ and assume $f(t)$ is piecewise continuous on $[0,\infty)$ and of exponential order α. Then, for $s > \alpha$,

$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n\frac{d^nF}{ds^n}(s).$$

A consequence of the above theorem is that if $f(t)$ is piecewise continuous and of exponential order, then its transform $F(s)$ has derivatives of all orders.

Example 3. Determine

(a) $\mathcal{L}\{t\cos 2t\mathrm{e}^{-t}\}$,

(b) $\mathcal{L}\{t^2\sin 4t\}$.

Properties of Laplace transform

$$\mathcal{L}\{f+g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\}$$
$$\mathcal{L}\{cf\} = c\mathcal{L}\{f\} \text{ for any constant } c$$
$$\mathcal{L}\{\mathrm{e}^{at}f\}(s) = F(s-a)$$
$$\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$$
$$\mathcal{L}\{f''\}(s) = s^2\mathcal{L}\{f\}(s) - sf(0) - f'(0)$$
$$\mathcal{L}\{f^{(n)}\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0)$$
$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n\frac{d^n}{ds^n}(\mathcal{L}\{f(t)\})(s)$$