## Chapter 7. Laplace Transforms.

### Section 7.4 **Inverse Laplace Transform.**

**Definition 1.** Given a function $F(s)$, if there is a function $f(t)$ that is continuous on $[0,\infty)$ and satisfies

$$\mathcal{L}\{f\}(s) = F(s),$$

then we say that $f(t)$ is the **inverse Laplace transform** of $F(s)$ and employ the notation $f(t) = \mathcal{L}^{-1}\{F\}(t)$.

### Table of inverse Laplace transform

| $F(s)$ | $f(t) = \mathcal{L}^{-1}\{F\}(t)$ |
|---|---|
| $\frac{1}{s}, \quad s > 0$ | $1$ |
| $\frac{1}{s-a}, \quad s > a$ | $e^{at}$ |
| $\frac{(n-1)!}{s^n}, \quad s > 0$ | $t^{n-1}, \quad n = 1, 2, ...$ |
| $\frac{b}{s^2+b^2}, \quad s > 0$ | $\sin bt$ |
| $\frac{s}{s^2+b^2}, \quad s > 0$ | $\cos bt$ |
| $\frac{(n-1)!}{(s-a)^n}, \quad s > a$ | $e^{at}t^{n-1}, \quad n = 1, 2, ...$ |
| $\frac{b}{(s-a)^2+b^2}, \quad s > a$ | $e^{at}\sin bt$ |
| $\frac{s-a}{(s-a)^2+b^2}, \quad s > a$ | $e^{at}\cos bt$ |

**Example 1.** Determine the inverse Laplace transform of the given function.

(a) $F(s) = \frac{2}{s^3}$.

SOLUTION. $\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} = \mathcal{L}^{-1}\left\{\frac{2!}{s^3}\right\} = t^2$

(b) $F(s) = \frac{2}{s^2+4}$.

SOLUTION. $\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\} = \sin 2t$.

(c) $F(s) = \frac{s+1}{s^2+2s+10}$.

SOLUTION. $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+2s+10}\right\} = \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+9}\right\} = \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+3^2}\right\} = e^{-t}\cos 3t$.

**Theorem 1. (linearity of the inverse transform)** Assume that $\mathcal{L}^{-1}\{F\}$, $\mathcal{L}^{-1}\{F_1\}$, and $\mathcal{L}^{-1}\{F_2\}$ exist and are continuous on $[0,\infty)$ and $c$ is any constant. Then

$$\mathcal{L}^{-1}\{F_1 + F_2\} = \mathcal{L}^{-1}\{F_1\} + \mathcal{L}^{-1}\{F_2\}$$
$$\mathcal{L}^{-1}\{cF\} = c\mathcal{L}^{-1}\{F\}.$$

**Example 2.** Determine $\mathcal{L}^{-1}\left\{\frac{3}{(2s+5)^3} + \frac{2s+16}{s^2+4s+13} + \frac{3}{s^2+4s+8}\right\}$.

SOLUTION. $\mathcal{L}^{-1}\left\{\frac{3}{(2s+5)^3}+\frac{2s+16}{s^2+4s+13}+\frac{3}{s^2+4s+8}\right\}=$

$$=\mathcal{L}^{-1}\left\{\frac{3}{(2s+5)^3}\right\}+\mathcal{L}^{-1}\left\{\frac{2s+16}{s^2+4s+13}\right\}+\mathcal{L}^{-1}\left\{\frac{3}{s^2+4s+8}\right\}=$$

$$=3\mathcal{L}^{-1}\left\{\frac{1}{2^3(s+\frac{5}{2})^3}\right\}+2\mathcal{L}^{-1}\left\{\frac{s+8}{(s+2)^2+9}\right\}+3\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+4}\right\}=$$

$$=\frac{3}{8}\mathcal{L}^{-1}\left\{\frac{1}{(s+\frac{5}{2})^3}\right\}+2\mathcal{L}^{-1}\left\{\frac{(s+2)+6}{(s+2)^2+3^2}\right\}+3\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+2^2}\right\}=$$

$$=\frac{3}{8\cdot 2}\mathcal{L}^{-1}\left\{\frac{2}{(s+\frac{5}{2})^3}\right\}+2\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+3^2}\right\}+2\mathcal{L}^{-1}\left\{\frac{2\cdot 3}{(s+2)^2+3^2}\right\}+\frac{3}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s+2)^2+2^2}\right\}=$$

$$=\frac{3}{16}\mathrm{e}^{-\frac{5}{2}t}t^2+2\mathrm{e}^{-2t}\cos 3t+4\mathrm{e}^{-2t}\sin 3t+\frac{3}{2}\mathrm{e}^{-2t}\sin 2t=$$

$$=\frac{3}{16}\mathrm{e}^{-\frac{5}{2}t}t^2+\mathrm{e}^{-2t}(2\cos 3t+4\sin 3t+\frac{3}{2}\sin 2t)$$

**Example 3.** Determine $\mathcal{L}^{-1}\{F\}$, where

(a) $F(s)=\frac{s^2-26s-47}{(s-1)(s+2)(s+5)}$,

SOLUTION. We begin by finding the partial fraction expantion for $F(s)$. The denominator consists of three linear factors, so the expantion has the form

$$\frac{s^2-26s-47}{(s-1)(s+2)(s+5)}=\frac{A}{s-1}+\frac{B}{s+2}+\frac{C}{s+5},$$

where numbers $A$, $B$, and $C$ to be determined.

$$\frac{s^2-26s-47}{(s-1)(s+2)(s+5)}=\frac{A}{s-1}+\frac{B}{s+2}+\frac{C}{s+5}=$$

$$=\frac{A(s+2)(s+5)+B(s-1)(s+5)+C(s-1)(s+2)}{(s-1)(s+2)(s+5)}.$$

So, we have that

$$s^2-26s-47=A(s+2)(s+5)+B(s-1)(s+5)+C(s-1)(s+2).$$

We can find $A$, $B$, and $C$ plugging $s=1$, $s=-2$, and $s=-5$ into last equality.

$$\begin{array}{ll} s=1: & 1-26-47=A(1+2)(1+5), \\ s=-2: & 4-26(-2)-47=B(-2-1)(-2+5), \\ s=-5: & 25-26(-5)-47=C(-5-1)(-5+2). \end{array}$$

So, $A=-4$, $B=-1$, $C=6$. Thus,

$$\frac{s^2-26s-47}{(s-1)(s+2)(s+5)}=-\frac{4}{s-1}-\frac{1}{s+2}+\frac{6}{s+5},$$

$$\mathcal{L}^{-1}\left\{\frac{s^2-26s-47}{(s-1)(s+2)(s+5)}\right\} = \mathcal{L}^{-1}\left\{-\frac{4}{s-1} - \frac{1}{s+2} + \frac{6}{s+5}\right\} =$$

$$= \mathcal{L}^{-1}\left\{-\frac{4}{s-1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \mathcal{L}^{-1}\left\{\frac{6}{s+5}\right\} = -4\mathrm{e}^{t} - \mathrm{e}^{-2t} + 6\mathrm{e}^{-5t}.$$

(b) $F(s) = \frac{3s^2+5s+3}{s^4+s^3}$,

SOLUTION. We begin by finding the partial fraction expantion for $F(s)$.

$$F(s) = \frac{3s^2+5s+3}{s^4+s^3} = \frac{3s^2+5s+3}{s^3(s+1)}.$$

Since $s$ is repeated factor with multiplicity 3 and $s+1$ is nonrepeated linear factor, the expantion has the form

$$\frac{3s^2+5s+3}{s^3(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s+1} =$$

$$= \frac{As^2(s+1) + Bs(s+1) + C(s+1) + Ds^3}{s^3(s+1)}.$$

Then

$$3s^2 + 5s + 3 = As^2(s+1) + Bs(s+1) + C(s+1) + Ds^3.$$

Setting $s = 0$ gives $C = 3$. Similarly, setting $s = -1$ gives $3 - 5 + 3 = -D$ or $D = -1$. Finally, to find $A$ and $B$ we pick two different values for $s$, say $s = 1$ and $s = 2$.

$$\begin{aligned} s = 1: &\quad 3+5+3 = 2A + 2B + 2C + D = 2A + 2B + 6 - 1 = 2A + 2B + 5, \\ s = 2: &\quad 12 + 10 + 3 = 12A + 6B + 3C + 8D = 12A + 6B + 18 - 8 = 12A + 6B + 10. \end{aligned}$$

We can determine $A$ and $B$ from the following system

$$\begin{cases} A + B = 3 \\ 4A + 2B = 5 \end{cases}$$

Solving the system gives $A = -1/2$, $B = 7/2$. Thus

$$\frac{3s^2+5s+3}{s^3(s+1)} = -\frac{1/2}{s} + \frac{7/2}{s^2} + \frac{3}{s^3} - \frac{1}{s+1},$$

$$\mathcal{L}^{-1}\left\{\frac{3s^2+5s+3}{s^3(s+1)}\right\} = \mathcal{L}^{-1}\left\{-\frac{1/2}{s} + \frac{7/2}{s^2} + \frac{3}{s^3} - \frac{1}{s+1}\right\} = -\frac{1}{2} + \frac{7}{2}t + \frac{1}{2}t^2 - \mathrm{e}^{-t}.$$