Chapter 7. Laplace Transforms.

Example 1. Find $\mathcal{L}^{-1}\left\{\dfrac{-5s-36}{(s+2)(s^2+9)}\right\}$.

SOLUTION. Since $s+2$ is nonrepeated linear factor and s^2+9 is the irreducible quadratic factor, the partial fraction expansion has the form

$$\frac{-5s-36}{(s+2)(s^2+9)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+9} = \frac{A(s^2+9)+(Bs+C)(s+2)}{(s+2)(s^2+9)}.$$

$$-5s-36 = A(s^2+9)+(Bs+C)(s+2).$$

Let' put in last equation $s=-2$, $s=0$, and $s=1$.

$$\begin{aligned} s=-2: &\quad -26 = 13A \\ s=0: &\quad -36 = 9A+2C \\ s=1: &\quad -41 = 9A+3B+3C \end{aligned}$$

Thus, $A=-2$, $B=2$, and $C=-9$. So,

$$\frac{-5s-36}{(s+2)(s^2+9)} = -\frac{2}{s+2} + \frac{2s-9}{s^2+9}.$$

$$\mathcal{L}^{-1}\left\{\frac{-5s-36}{(s+2)(s^2+9)}\right\} = -2\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \mathcal{L}^{-1}\left\{\frac{2s-9}{s^2+9}\right\} =$$

$$= -2e^{-2t} + 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} - 3\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\} = -2e^{-2t} + 2\cos 3t - 3\sin 3t.$$

Section 7.5 **Solving initial value problems.**

To solve an initial value problem:

(a) Take the Laplace transform of both sides of the equation.

(b) Use the properties of the Laplace transform and the initial conditions to obtain an equation for the Laplace transform of the solution and then solve this equation for the transform.

(c) Determine the inverse Laplace transform of the solution.

Important formulas:

$$\begin{aligned} &\mathcal{L}\{y'\}(s) = s\mathcal{L}\{y\}(s) - y(0) \\ &\mathcal{L}\{y''\}(s) = s^2\mathcal{L}\{y\}(s) - sy(0) - y'(0) \end{aligned}$$

Example 2. Solve the initial value problem

$$y'' + 6y' + 5y = 12e^t, \quad y(0) = -1, \quad y'(0) = 7.$$

Take the Laplace transform of both sides of the equation.

$$\mathcal{L}\{y'' + 6y' + 5y\} = \mathcal{L}\{12e^t\}$$

$$\mathcal{L}\{y'' + 6y' + 5y\} = \mathcal{L}\{y''\} + 6\mathcal{L}\{y'\} + 5\mathcal{L}\{y\}.$$

Let $\mathcal{L}\{y\}(s) = Y(s)$, then

$$\mathcal{L}\{y'\}(s) = sY(s) - y(0) = sY(s) + 1$$

$$\mathcal{L}\{y''\}(s) = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) + s - 7$$

$$\mathcal{L}\{y'' + 6y' + 5y\} = \mathcal{L}\{y''\} + 6\mathcal{L}\{y'\} + 5\mathcal{L}\{y\} = (s^2 + 6s + 5)Y(s) + s - 1.$$

Note, that coefficient of $Y(s)$ is the corresponding auxiliary equation to the equation $y'' + 6y' + 5y = 0$.

Since

$$\mathcal{L}\{12\mathrm{e}^t\} = \frac{12}{s-1},$$

we have

$$(s^2 + 6s + 5)Y(s) + s - 1 = \frac{12}{s-1}$$

or

$$Y(s) = \frac{12 - (s-1)^2}{(s-1)(s^2+6s+5)} = \frac{12 - (s-1)^2}{(s-1)(s+1)(s+5)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+5} =$$

$$= \frac{A(s+1)(s+5) + B(s-1)(s+5) + C(s-1)(s+1)}{(s-1)(s+1)(s+5)}.$$

$$12 - (s-1)^2 = A(s+1)(s+5) + B(s-1)(s+5) + C(s-1)(s+1).$$

To determine A, B, and C let's plug $s = 1$, $s = -1$, and $s = -5$ in last equation.

$$\begin{array}{ll} s = 1: & 12 = 12A \\ s = -1 & 8 = -8B \\ s = -5: & -24 = 24C \end{array}$$

Thus, $A = 1$, $B = -1$, $C = -1$. Since

$$Y(s) = \frac{1}{s-1} - \frac{1}{s+1} - \frac{1}{s+5},$$

the solution to the given initial problem is

$$y(t)\mathcal{L}^{-1}\{Y(s)\} = \mathrm{e}^t - \mathrm{e}^{-t} - \mathrm{e}^{-5t}.$$

Example 3. Solve the initial value problem

$$y'' - 2y' + y = 6t - 2, \quad y(-1) = 3, \ \ y'(-1) = 7.$$

SOLUTION. The initial conditions are given at $t = -1$ not at $t = 0$. Let

$$u = t + 1,$$

$u = 0$ when $t = -1$.

Replacing t by u in the differential equation, we have

$$y''(u) - 2y'(u) + y(u) = 6u - 8$$

and the initial conditions become

$$y(0) = 3, \quad y'(0) = 7.$$

Because the initial conditions now are given at the origin, the Laplace transform method is applicable.

$$\mathcal{L}\{y''(u) - 2y'(u) + y(u)\} = \mathcal{L}\{6u - 8\}$$

Let $\mathcal{L}\{y(u)\}(s) = Y(s)$, then

$$\mathcal{L}\{y'\}(s) = sY(s) - y(0) = sY(s) - 3$$

$$\mathcal{L}\{y''\}(s) = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 3s - 7$$

$$\mathcal{L}\{y'' - 2y' + y\} = \mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = (s^2 - 2s + 1)Y(s) - 3s - 1.$$

Since

$$\mathcal{L}\{6u - 8\} = \frac{6}{s^2} - \frac{8}{s} = \frac{6 - 8s}{s^2},$$

we have

$$(s^2 - 2s + 1)Y(s) - 3s - 1 = \frac{6 - 8s}{s^2}$$

or

$$Y(s) = \frac{6 - 8s + s^2(3s + 1)}{s^2(s-1)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{(s-1)^2} =$$
$$= \frac{As(s-1)^2 + B(s-1)^2 + Cs^2(s-1) + Ds^2}{s^2(s-1)^2}.$$

$$6 - 8s + s^2(3s + 1) = As(s-1)^2 + B(s-1)^2 + Cs^2(s-1) + Ds^2.$$

To determine A, B, C, and D we set $s = 0$, $s = 1$, $s = -1$, and $s = 2$ in the last equation.

$$\begin{aligned} s = 0 &: \quad 6 = B \\ s = 1 &: \quad 2 = D \\ s = -1 & \quad 2A + C = 7 \\ s = 2 &: \quad A + 2C = 2 \end{aligned}$$

Thus, $A = 4$, $B = 6$, $C = -1$, $D = 2$. Since

$$Y(s) = \frac{4}{s} + \frac{6}{s^2} - \frac{1}{s-1} + \frac{2}{(s-1)^2},$$

the solution to the given initial problem is

$$y(t)\mathcal{L}^{-1}\{Y(s)\} = 4 + 6u - e^u + 2ue^u = 4 + 6(t+1) - e^{t+1} + 2(t+1)e^{t+1}.$$