Example 1.
(a) Determine for which values of \( m \) the function \( \varphi(x) = x^m \), where \( x \neq 0 \), is a solution to the equation

\[
3x^2 \frac{d^2 y}{dx^2} + 11x \frac{dy}{dx} - 3y = 0.
\]

SOLUTION We compute \( \varphi'(x) = mx^{m-1} \) and \( \varphi''(x) = m(m-1)x^{m-2} \). Substituting \( \varphi(x) \), \( \varphi'(x) \), and \( \varphi''(x) \) for \( y, y', \) and \( y'' \) in the given equation yields

\[
3x^2 \cdot m(m-1)x^{m-2} + 11x \cdot mx^{m-1} - 3x^m = x^m(3m(m-1) + 11m - 3) = 0.
\]

Since \( x \neq 0 \),

\[
3m(m-1) + 11m - 3 = 3m^2 + 8m - 3 = 0.
\]

This quadratic equation has two solutions \( m_1 = 1/3 \), and \( m_2 = -3 \). Thus, \( \varphi(x) = x^{1/3} = \sqrt[3]{x} \), and \( \varphi(x) = x^{-3} = \frac{1}{x^3} \) are solutions for the given equation.

(b) Show that \( e^{xy} + y = x - 1 \) is an implicit solution to

\[
\frac{dy}{dx} = \frac{e^{-xy} - y}{e^{-xy} + x}.
\]

SOLUTION Let's implicitly differentiate the equation \( e^{xy} + y = x - 1 \) with respect to \( x \)

\[
\frac{d}{dx}(e^{xy} + y) = \frac{d}{dx}(x - 1),
\]

\[
e^{xy} \left( y + x \frac{dy}{dx} \right) + \frac{dy}{dx} = 1,
\]

or

\[
\frac{dy}{dx} \left( 1 + xe^{xy} \right) = 1 - ye^{xy},
\]

which is equivalent to the given equation.

Definition 1. By an initial value problem for an \( n \)-th-order differential equation

\[
F \left( x, y, \frac{dy}{dx}, \cdots, \frac{d^n y}{dx^n} \right) = 0
\]

we mean: Find a solution to the differential equation on an interval \( I \) that satisfies at \( x_0 \) the \( n \) initial conditions:
\[ y(x_0) = y_0, \]
\[ \frac{dy}{dx}(x_0) = y_1, \]
\[ \vdots \]
\[ \frac{d^{n-1}y}{dx^{n-1}}(x_0) = y_{n-1}, \]
where \( x_0 \in I \) and \( y_0, y_1, \ldots, y_{n-1} \) are given constants.

In case of a first-order equation \( F(x, y, \frac{dy}{dx}) \), the initial conditions reduce to the single requirement
\[ y(x_0) = y_0, \]
and in the case of a second-order equation, the initial conditions have the form
\[ y(x_0) = y_0, \quad \frac{dy}{dx} = y_1. \]

Example 2. As shown before, the function \( \varphi(x) = C_1 \cos 5x + C_2 \sin 5x \) is a solution to
\[ y'' + 25y = 0 \]
for any choice of the constants \( C_1 \) and \( C_2 \). Determine \( C_1 \) and \( C_2 \) so that the initial conditions
\[ y(0) = \pi \quad \text{and} \quad \frac{dy}{dx}(0) = 1 \]
are satisfied.

SOLUTION To determinate the constants \( C_1 \) and \( C_2 \), we first compute \( \varphi'(x) \) to get
\[ \varphi'(x) = -5C_1 \sin 5x + 5C_2 \cos 5x. \]
Substituting in our initial conditions gives the following equations:
\[ \varphi(0) = C_1 \cdot 1 + C_2 \cdot 0 = C_1 = \pi, \]
\[ \varphi'(0) = -5C_1 \cdot 0 + 5C_2 \cdot 1 = 5C_2 = 1. \]
We find \( C_1 = \pi, C_2 = 1/5 \). Hence, the solution of the initial value problem is \( \varphi(x) = \pi \cos 5x + \frac{1}{5} \sin 5x. \)

We now state an existence and uniqueness theorem for first-order initial value problems.

Theorem 1. Given the initial value problem
\[ \frac{dy}{dx} = f(x, y), \quad y(0) = y_0, \]
assume that \( f \) and \( \partial f/\partial y \) are continuous functions in a rectangle
\[ R = \{(x, y) : a < x < b, c < y < d\}. \]
that contains the point \((x_0, y_0)\). Then the initial value problem has a unique solution \(\varphi(x)\) in some interval \(x_0 - \delta < x < x_0 + \delta\), where \(\delta\) is a positive number.

Theorem 1 tells us two things. First, when an equation satisfies the hypotheses of Theorem 1, we are assured that a solution to the initial problem exists. Second, when the hypotheses are satisfied, there is a unique solution to the initial value problem. Notice that theorem works only in some neighborhood \((x_0 - \delta, x_0 + \delta)\)

Example 3. For the initial value problem

\[
\frac{dy}{dx} = x^3 - y^3, \quad y(0) = 6,
\]

does Theorem 1 imply the existence of a unique solution?

SOLUTION Here \(f(x, y) = x^3 - y^3\) and \(\frac{\partial f}{\partial y} = 3y^2\). Both of these functions are continuous in \(R = \{(x, y) : -\infty < x < +\infty, -\infty < y < +\infty\}\), so the hypotheses of Theorem 1 are satisfied. It the follows from Theorem 1 that the given initial value problem has a unique solution in an interval about \(x = 0\) of the form \((-\delta, \delta)\), where \(\delta\) is some positive number.

1.3 Direction Fields

For practical reasons we may need to know the value of the solution at a certain point, or the intervals where the solution is increasing, or the point where the solution attains the maximum value. Certainly, knowing an explicit representation for the solution would be a help in answering these questions. However, for many of the differential equations it will be impossible to find such a formula. Thus, we must rely on other methods to analyze or approximate the solution.

One technique that is useful in graphing the solutions to a first-order differential equation is to sketch the direction field for the equation. To describe this method, we need to make a general observation. Namely, a first-order equation
\[ \frac{dy}{dx} = f(x, y) \]

specifies a slope at each point in the xy-plane where \( f \) is defined.

A plot of short line segments drawn at various points in the xy-plane showing the slope of the solution curve there is called a **direction field** for the differential equation. Because the direction field gives the “flow of solutions”, it facilitates the drawing of any particular solution (such as the solution to an initial value problem).

**Figure 1.** Direction field for \( \frac{dy}{dx} = x^2 - y \)

**Figure 2.** Solutions to \( \frac{dy}{dx} = x^2 - y \)