Section 1.2 **Solutions and Initial Value Problems**

Example 1.

(a) Determine for which values of m the function $\varphi(x) = x^m$, where $x \neq 0$, is a solution to the equation

$$3x^2 \frac{d^2y}{dx^2} + 11x\frac{dy}{dx} - 3y = 0.$$

SOLUTION We compute $\varphi'(x) = mx^{m-1}$ and $\varphi''(x) = m(m-1)x^{m-2}$. Substituion of $\varphi(x)$, $\varphi'(x)$, and $\varphi''(x)$ for y, y', and y'' in given equation yelds

$$3x^2 \cdot m(m-1)x^{m-2} + 11x \cdot mx^{m-1} - 3x^m = x^m(3m(m-1) + 11m - 3) = 0.$$

Since $x \neq 0$,

$$3m(m-1) + 11m - 3 = 3m^2 + 8m - 3 = 0.$$

This quadratic equation has two solutions $m_1 = 1/3$, and $m_2 = -3$. Thus, $\varphi(x) = x^{1/3} = \sqrt[3]{x}$, and $\varphi(x) = x^{-3} = \frac{1}{x^3}$ are solutions for the given equation.

(b) Show that $\mathrm{e}^{xy} + y = x - 1$ is an implicit solution to

$$\frac{dy}{dx} = \frac{\mathrm{e}^{-xy} - y}{\mathrm{e}^{-xy} + x}.$$

SOLUTION Lets implicitly differentiate the equation $\mathrm{e}^{xy} + y = x - 1$ with rspect to x

$$\frac{d}{dx}(\mathrm{e}^{xy} + y) = \frac{d}{dx}(x-1),$$

$$\mathrm{e}^{xy}\left(y + x\frac{dy}{dx}\right) + \frac{dy}{dx} = 1,$$

$$\frac{dy}{dx}(1 + x\mathrm{e}^{xy}) = 1 - y\mathrm{e}^{xy},$$

or

$$\frac{dy}{dx} = \frac{1 - y\mathrm{e}^{xy}}{1 + x\mathrm{e}^{xy}},$$

which is equivalent to the given equation.

Definition 1. By an *initial value problem* for an nth-order differential equation

$$F\left(x, y, \frac{dy}{dx}, \cdots, \frac{d^ny}{dx^n}\right) = 0$$

we mean: Find a solution to the differential equation on an interval I that satisfies at x_0 the n initial conditions:

$$y(x_0) = y_0,$$
$$\frac{dy}{dx}(x_0) = y_1,$$
$$\vdots$$
$$\frac{d^{n-1}y}{dx^{n-1}}(x_0) = y_{n-1},$$
where $x_0 \in I$ and y_0, y_1,...,y_{n-1} are given constants.

In case of a first-order equation $F\left(x, y, \frac{dy}{dx}\right)$, the initial conditions reduce to the single requirement
$$y(x_0) = y_0,$$
and in the case of a second-order equation, the initial conditions have the form
$$y(x_0) = y_0, \qquad \frac{dy}{dx} = y_1.$$

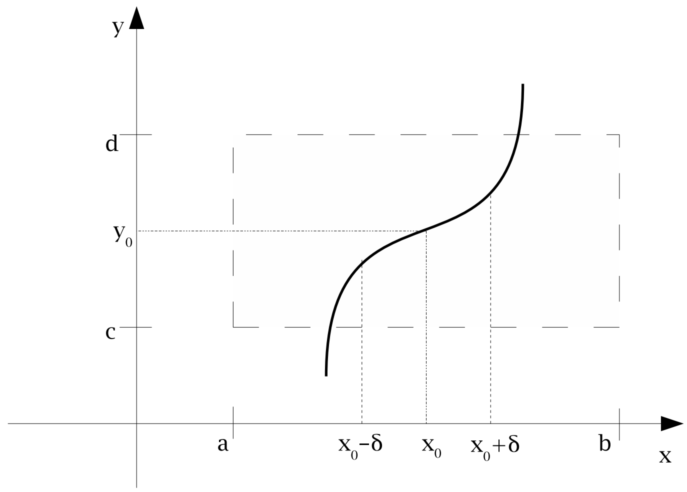
Example 2. As shown before, the function $\varphi(x) = C_1 \cos 5x + C_2 \sin 5x$ is a solution to $y'' + 25y = 0$ for any choice of the constants C_1 and C_2. Determine C_1 and C_2 so that the initial conditions
$$y(0) = \pi \quad and \quad \frac{dy}{dx}(0) = 1$$
are satisfied.

SOLUTION To determinate the constants C_1 and C_2, we first compute $\varphi'(x)$ to get
$$\varphi'(x) = -5C_1 \sin 5x + 5C_2 \cos 5x.$$
Substituting in our initial conditions gives the following equations:
$$\varphi(0) = C_1 \cdot 1 + C_2 \cdot 0 = C_1 = \pi,$$
$$\varphi'(0) = -5C_1 \cdot 0 + 5C_2 \cdot 1 = 5C_2 = 1.$$
We find $C_1 = \pi$, $C_2 = 1/5$. Hence, the sotion of the initial value problem is $\varphi(x) = \pi \cos 5x + \frac{1}{5} \sin 5x$.

We now state an existence and uniqueness theorem for first-order initial value problems.

Theorem 1. Given the initial value problem
$$\frac{dy}{dx} = f(x, y), \qquad y(0) = y_0,$$
assume that f and $\partial f / \partial y$ are continuous functions in a rectangle
$$R = \{(x, y) : a < x < b, c < y < d\}$$

that contains the point (x_0, y_0). Then the initial value problem has a unique solution $\varphi(x)$ in some interval $x_0 - \delta < x < x_0 + \delta$, where δ is a positive number.



Theorem 1 tells us two things. First, when an equation satisfies the hypotheses of Theorem 1, we are assured that a solution to the initial problem exists. Second, when the hypotheses are satisfied, there is a **unique** solution to the initial value problem. Notice that theorem works only in some neighborhood $(x_0 - \delta, x_0 + \delta)$

Example 3. For the initial value problem

$$\frac{dy}{dx} = x^3 - y^3, \qquad\qquad y(0) = 6,$$

does Theorem 1 imply the existence of a unique solution?

SOLUTION Here $f(x, y) = x^3 - y^3$ and $\frac{\partial f}{\partial y} = 3y^2$. Both of these functions are continuous in $R = \{(x, y) : -\infty < x < +\infty, -\infty < y < +\infty\}$, so the hypotheses of Theorem 1 are satisfied. It the follows from Theorem 1 that the given initial value problem has a unique solution in an interval about $x = 0$ of the form $(-\delta, \delta)$, where δ is some positive number.

1.3 **Direction Fields**

For practical reasons we may need to know the value of the solution at a certain point, or the intervals where the solution is increasing, or the point where the solution attains the maximum value. Certainly, knowing an explicit representation for the solution would be a help in answering these questions. However, for many of the differential equations it will be impossible to find such a formula. Thus, we must rely on other methods to analyze or approximate the solution.

One technique that is useful in graphing the solutions to a first-order differential equation is to sketch the direction field for the equation. To describe this method, we need to make a general observation. Namely, a first-order equation

$$\frac{dy}{dx} = f(x, y)$$

specifies a slope at each point in the xy-plane where f is defined.

A plot of short line segments drawn at various points in the xy-plane showing the slope of the solution curve there is called a **direction field** for the differential equation. Because the direction field gives the "flow of solutions", it facilitates the drawing of any particular solution (such as the solution to an initial value problem).

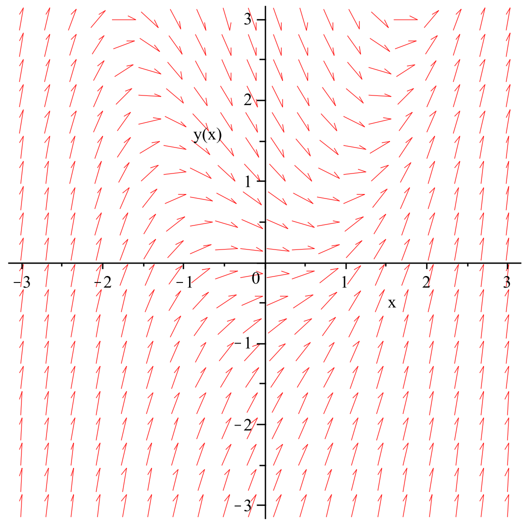


Figure 1. Direction field for $\frac{dy}{dx} = x^2 - y$

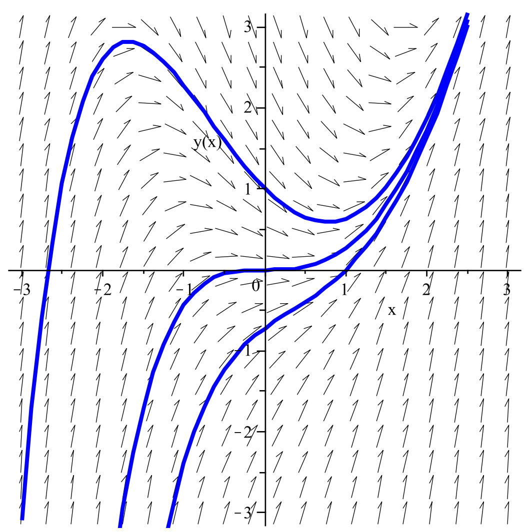


Figure 2. Solutions to $\frac{dy}{dx} = x^2 - y$