Example 1. Draw the isoclines with their direction markers and sketch several solution curves, including the the curve satisfying the given initial condition

$$y' = 2x^2 - y, \quad y(0) = 0.$$

SOLUTIONS

The isoclines for the given equations are the parabolas $2x^2 - y = C$, here C is an arbitrary constant.

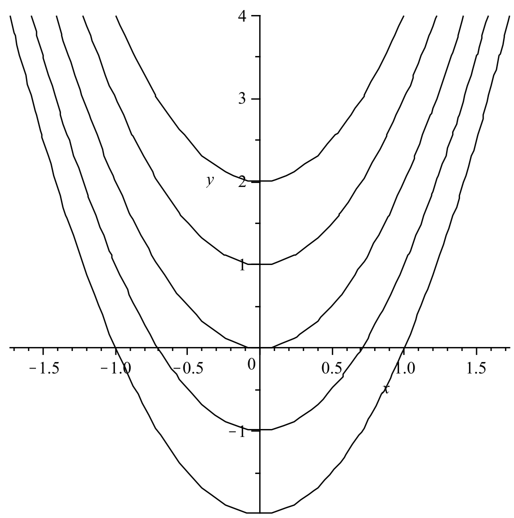

Figure 1. Isoclines for $y' = 2x^2 - y$

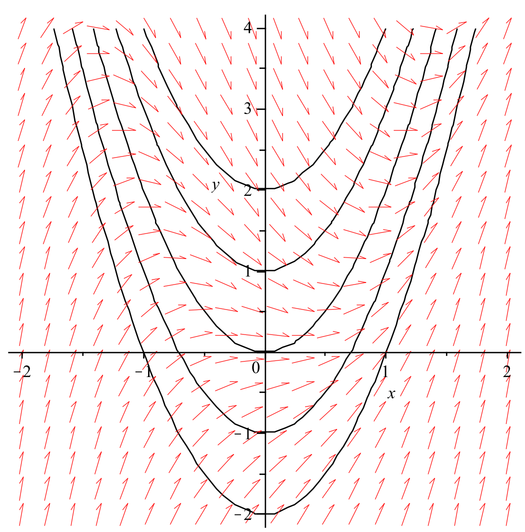

Figure 2. Direction field for $y' = 2x^2 - y$

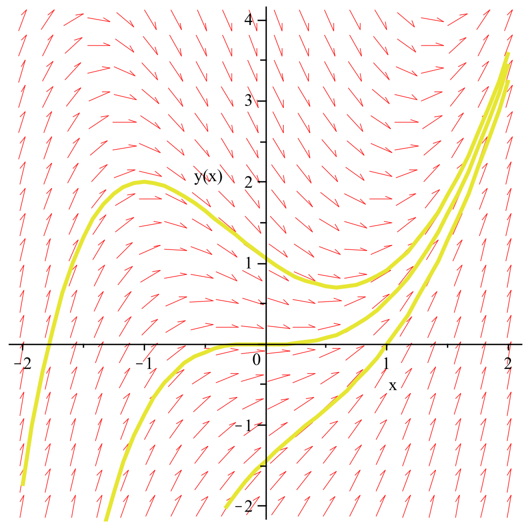

Figure 3. Solutions to $y' = 2x^2 - y$

Section 1.4 **The Approximation Method of Euler**

Euler's method (or the tangent line method) is a procedure for constructing approximate solutions to an initial value problem for a first-order differential equation

$$\begin{aligned} y' &= f(x,y), \\ y(x_0) &= y_0. \end{aligned} \tag{1}$$

The main idea of this method is to construct a polygonal (broken line) approximation to the solutions of the problem (1).

Assume that the the problem (1) has a unique solution $\varphi(x)$ in some interval centered at x_0. Let h be a fixed positive number (called the *step size*) and consider the equally spaced points

$$x_n := x_0 + nh, \qquad n = 0, 1, 2, \ldots$$

The construction of values y_n that approximate the solution values $\varphi(x_n)$ proceeds as follows. At the point (x_0, y_0), the slope of the solution to (1) is given by $dy/dx = f(x_0, y_0)$. Hence, the tangent line to the curve $y = \varphi x$ at the initial point (x_0, y_0) is

$$y - y_0 = f(x_0, y_0)(x - x_0), \quad \text{or}$$

$$y = y_0 + f(x_0, y_0)(x - x_0).$$

Using the tangent line to approximate φx, we find that for the point $x_1 = x_0 + h$

$$\varphi(x_1) \approx y_1 := y_0 + f(x_0, y_0)(x - x_0).$$

Next, starting at the point (x_1, y_1), we construct the line with slope equal to $f(x_1, y_1)$. If we follow the line in stepping from x_1 to $x_2 = x_1 + h$, we arrive at the approximation

$$\varphi(x_2) \approx y_2 := y_1 + f(x_1, y_1)(x - x_1).$$

Repeating the process, we get

$$\varphi(x_3) \approx y_3 := y_2 + f(x_2, y_2)(x - x_2),$$
$$\varphi(x_4) \approx y_4 := y_3 + f(x_3, y_3)(x - x_3), \text{etc.}$$

This simple procedure is **Euler's method** and can be summarized by the recursive formulas

$$x_{n+1} := x_0 + (n+1)h, \tag{2}$$

$$y_{n+1} := y_n + f(x_n, y_n)(x - x_n), \quad n = 0, 1, 2, \ldots \tag{3}$$

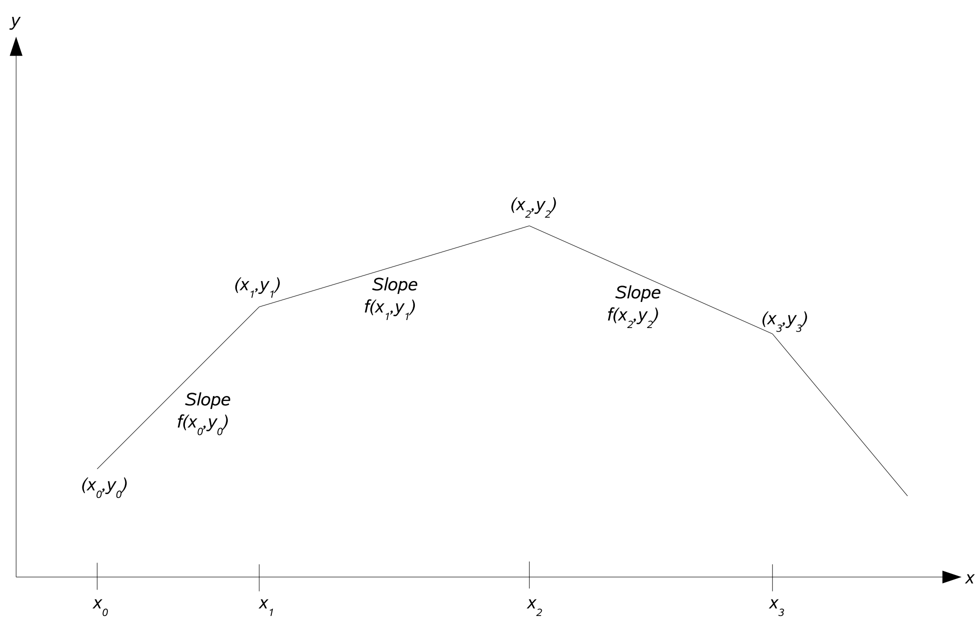

Figure 1. Polygonal-line approximation given by Euler's method