Chapter 2. First-Order Differential Equations

In this chapter we learn how to obtain solutions for some specific types of first-order equations. We begin by studying separable equations, then the linear equations.

Section 2.1 Introduction: Motion Of a Falling Body (Home reading)

Sections 2.4–2.6 are skipped.

Section 2.2 **Separable Equations**

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

Sometimes function $f(x, y)$ can be represented as a product of two functions, one of which depends ONLY on x, another depends ONLY on y, or $f(x, y) = g(x)h(y)$. Then

$$\frac{dy}{dx} = g(x)h(y). \tag{2}$$

Lets multiply equation (2) by $\frac{dx}{h(y)}$, then

$$\frac{dy}{h(y)} = g(x)dx,$$

$$\int \frac{dy}{h(y)} = \int g(x)dx,$$

Thus, the solution to an equation (2) is

$$H(y) = G(x) + C,$$

here $H(y)$ is an antiderivative of $1/h(y)$, $G(x)$ is an antiderivative of $g(x)$, C is a constant.

Example 1. Determine whether the given equation is separable.

(a) $(x - 2y)^2 y' = 2$

The ANSWER is NO.

(b) $y^4 \mathrm{e}^y + (x^3 + 1)y' = y'(x^3 + 1)\mathrm{e}^{2y}$

The ANSWER is YES, because

$$y' = (x^3 + 1)\frac{y^4 \mathrm{e}^y}{\mathrm{e}^{2y} - 1} = g(x)p(y).$$

(c) $yx \ln x dx - \sqrt{y} dy + x \ln x dx = 0$

The ANSWER is YES

$$x \ln x dx = \frac{\sqrt{y}}{1 + y}$$

(d) $y' = \cot^2\left(\frac{x}{2} + y - 1\right) + \frac{1}{2}$

The ANSWER is NO.

Example 2. Solve the equations/initial value problems:

(a) $xydx + (x+1)dy = 0$

SOLUTION Lets separate variables and rewrite the equation in the form

$$\frac{dy}{y} = -\frac{x}{x+1}dx.$$

Integrating, we have

$$\int \frac{dy}{y} = -\int \frac{x}{x+1}dx$$

$$\ln y = -x + \ln(x+1) + C,$$

and solving this last equation for y gives

$$y = \mathrm{e}^{-x+\ln(x+1)+C} = C_1(x+1)\mathrm{e}^{-x},$$

where $C_1 = \mathrm{e}^C$.

(b) $(x^2-1)y' + 2xy^2 = 0, \quad y(0) = 1$

SOLUTION Separating the variables and integrating gives

$$\frac{dy}{y^2} = -\frac{2xdx}{x^2-1},$$

$$\int \frac{dy}{y^2} = -\int \frac{2xdx}{x^2-1},$$

$$\frac{1}{y} = \ln|x^2-1| + C,$$

$$y = \frac{1}{\ln|x^2-1| + C}.$$

Putting $x = 0$ in solution gives

$$y(0) = \frac{1}{C} = 1,$$

and so $C = 1$. Thus, the solution to the initial value problem is

$$y = \frac{1}{\ln|x^2-1| + 1}.$$

(c) $xydx - \sqrt{x^2+1}\ln^2 ydy = 0$

SOLUTION Separating the variables and integrating gives

$$\frac{\ln^2 ydy}{y} = \frac{xdx}{\sqrt{x^2+1}},$$

$$\int \frac{\ln^2 ydy}{y} = \int \frac{xdx}{\sqrt{x^2+1}},$$

$$\frac{\ln^3 y}{3} = \sqrt{x^2+1} + C,$$

Thus, the implicit solution to the equation is

$$\ln^3 y = 3\sqrt{x^2+1} + C_1,$$

where $C_1 = 3C$.

(d) $x\cos^2 y dx - \mathrm{e}^x \sin 2y dy = 0, \quad y(0) = 0$

SOLUTION Separating the variables and integrating gives

$$\frac{\sin 2y dy}{\cos^2 y} = x\mathrm{e}^x dx,$$

$$2\frac{\sin y dy}{\cos y} = x\mathrm{e}^x dx,$$

$$2\int \frac{\sin y dy}{\cos y} = \int x\mathrm{e}^x dx,$$

$$-2\ln|\cos y| = (x-1)\mathrm{e}^x + C.$$

Thus, the implicit solution to the given equation is

$$\ln|\cos y| = \frac{1}{2}(1-x)\mathrm{e}^x + C_1,$$

where $C_1 = -1/2C$. Finally, we have to determine C_1 such that the initial condition is satisfied. Putting $x = 0$ and $y = 0$ in solution gives $0 = 1/2 + C_1$, so $C_1 = -1/2$. Thus, the solution to the initial value problem is

$$\ln|\cos y| = \frac{1}{2}(1-x)\mathrm{e}^x - \frac{1}{2}.$$