Chapter 2. First-Order Differential Equations

Example 1. Solve the equation

$$x^2y^2y' + 1 = y.$$

SOLUTION   Separating the variables and integrating gives

$$\frac{y^2}{y-1}dy = \frac{dx}{x},$$

$$\int \frac{y^2}{y-1}dy = \int \frac{dx}{x}.$$

So, the implicit solution to the equation is

$$\frac{y^2}{2} + y + \ln|y-1| = -\frac{1}{x} + C.$$

Section 2.3 **Linear Equations**

A **linear first-order equation** is an equation that can be expressed in the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = b(x), \tag{1}$$

where $a_0(x)$, $a_1(x)$, $b(x)$ depend only on x.

We will assume that $a_0(x)$, $a_1(x)$, $b(x)$ are continuous functions of x on an interval I.

For now, we are interested in those linear equations for which $a_1(x)$ is never zero on I. In that case we can rewrite (1) in the **standard form**

$$\frac{dy}{dx} + P(x)y = Q(x), \tag{2}$$

where $P(x) = a_0(x)/a_1(x)$ and $Q(x) = b(x)/a_1(x)$ are continuous on I.

There are two methods of solving linear first-order differential equations.

Method 1.

Equation (2) can be solved by finding an **integrating factor** $\mu(x)$ such that

$$\frac{d}{dx}[\mu y] = \frac{dy}{dx}\mu + \frac{d\mu}{dx}y = \mu Q(x).$$

Since,

$$\frac{dy}{dx} = Q(x) - P(x)y,$$

$$\frac{d}{dx}[\mu y] = \frac{dy}{dx}\mu + \frac{d\mu}{dx}y = (Q(x) - P(x)y)\mu + \frac{d\mu}{dx}y = \frac{d\mu}{dx}y - P(x)y\mu + \mu Q(x) =$$

$$\left(\frac{d\mu}{dx} - P(x)\mu\right)y + \mu Q(x) = \mu Q(x).$$

Clearly, this requires that μ satisfy

$$\frac{d\mu}{dx} - P(x)\mu = 0. \tag{3}$$

Lets find the solution to equation (3).

$$\frac{d\mu}{dx} - P(x)\mu = 0,$$

$$\frac{d\mu}{dx} = P(x)\mu,$$

$$\frac{d\mu}{\mu} = P(x)dx,$$

$$\int \frac{d\mu}{\mu} = \int P(x)dx,$$

$$\ln \mu = \int P(x)dx,$$

$$\mu = \exp\left[\int P(x)dx\right]. \tag{4}$$

With this choice for $\mu(x)$, equation (2) becomes

$$\frac{d}{dx}\left[\mu(x)y\right] = \mu(x)Q(x),$$

$$\mu(x)y = \int \mu(x)Q(x)dx + C,$$

$$y = \frac{1}{\mu}\int \mu(x)Q(x)dx + C, \tag{5}$$

here C is an arbitrary constant.
The solution (5) to equation (2) is called the **general solution**.

Example 2. Obtain the general solution to the equation

$$xy' - y = -\ln x.$$

SOLUTION To put this equation in standard form, we divide by x to obtain

$$y' - \frac{y}{x} = -\frac{\ln x}{x}.$$

Here $P(x) = -\frac{1}{x}$, $Q(x) = -\frac{\ln x}{x}$.
We can find the integrating factor $\mu(x)$ solving the equation

$$\frac{d\mu}{dx} + \frac{1}{x}\mu = 0.$$

Separating the variables and integrating gives

$$\frac{d\mu}{\mu} = -\frac{dx}{x},$$

$$\int \frac{d\mu}{\mu} = -\int \frac{dx}{x},$$

$$\ln \mu = -\ln x,$$

$$\mu(x) = \frac{1}{x}.$$

Now, we can find y from the equation

$$\frac{d}{dx}\left[-\frac{1}{x}y\right] = \frac{1}{x}\frac{\ln x}{x} = \frac{\ln x}{x^2}$$

Integrating gives

$$-\frac{y}{x} = \int \frac{\ln x}{x^2}dx = -\frac{\ln x}{x} - \frac{1}{x} + C.$$

Thus, the general solution to the given equation is

$$y(x) = \ln x + 1 + Cx.$$

Method 2 (variation of parameter).

Associated with equation $y' + P(x)y = Q(x)$ is the equation

$$y' + P(x)y = 0, \tag{6}$$

which is obtained from (2) by replacing $Q(x)$ with zero. We say that equation (2) is a **nonhomogeneous equation** and that (6) is the corresponding **homogeneous equation**.

We can obtain the general solution to the nonhomegeneous equation solving homogeneous equation

$$y' + P(x)y = 0.$$

The solution to homogeneous equation is

$$y_{\text{hom}}(x) = C \exp\left[-\int P(x)dx\right],$$

then the general solution to the nonhomogeneous equation is

$$y(x) = C(x) \exp\left[-\int P(x)dx\right], \tag{7}$$

where $C(x)$ is an unknown function that depends on x.

Since

$$y'(x) = C'(x)\exp\left[-\int P(x)dx\right] - C(x)P(x)\exp\left[-\int P(x)dx\right], \tag{8}$$

substitution of right parts of (8) and (7) for y and y' in nonhomogeneous equation gives

$$C'(x)\exp\left[-\int P(x)dx\right] - C(x)P(x)\exp\left[-\int P(x)dx\right] + P(x)C(x)\exp\left[-\int P(x)dx\right] = Q(x),$$

$$C'(x)\exp\left[-\int P(x)dx\right] = Q(x),$$

$$C'(x) = Q(x)\exp\left[\int P(x)dx\right],$$

$$C'(x) = \int Q(x)\exp\left[\int P(x)dx\right]dx + C_1,$$

where C_1 is an arbitrary constant. Thus, the general solution to nonhomogeneous equation is

$$y(x) = \exp\left[-\int P(x)dx\right]\int Q(x)\exp\left[\int P(x)dx\right]dx + C_1.$$

Example 3. Obtain the general solution to the equation

$$xy' - y = -\ln x$$

using Method 2.

SOLUTION. The corresponding homogeneous equation to the given equation is

$$xy' - y = 0.$$

Separating the variables and integrating gives

$$\frac{dy}{y} = \frac{dx}{x},$$

$$\int \frac{dy}{y} = \int \frac{dx}{x},$$

$$y_{\text{hom}} = Cx.$$

The general solution to the given nonhomogeneous equation is

$$y(x) = C(x)x,$$

where $C(x)$ is an unknown function.

$$y'(x) = C'(x)x + C(x).$$

Then

$$C'(x)x^2 + C(x)x - C(x)x = -\ln x,$$

$$C'(x) = -\frac{\ln x}{x^2},$$

$$C(x) = -\int \frac{\ln x}{x^2} dx = \frac{\ln x}{x} + \frac{1}{x} + C_1.$$

Thus, the general solution to the given nonhomogeneous equation is

$$y(x) = \left(\frac{\ln x}{x} + \frac{1}{x} + C_1 \right) x = \ln x + 1 + C_1 x.$$

Existence and uniqueness of solution

Theorem 1. Suppose $P(x)$ and $Q(x)$ are continuous on some interval I that contains the point x_0. Then for any choice of initial value y_0, there exists a unique solution $y(x)$ on I to the initial value problem

$$y' + P(x)y = Q(x), \qquad y(x_0) = y_0. \tag{9}$$

In fact, the solution is given by (5) for a suitable value of C.