Chapter 2. First-Order Differential Equations
Section 2.3 **Linear Equations**

A **linear first-order equation** is an equation that can be expressed in the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = b(x), \tag{1}$$

where $a_0(x)$, $a_1(x)$, $b(x)$ depend only on x.
We will assume that $a_0(x)$, $a_1(x)$, $b(x)$ are continuous functions of x on an interval I.
For now, we are interested in those linear equations for which $a_1(x)$ is never zero on I. In that case we can rewrite (1) in the **standard form**

$$\frac{dy}{dx} + P(x)y = Q(x), \tag{2}$$

where $P(x) = a_0(x)/a_1(x)$ and $Q(x) = b(x)/a_1(x)$ are continuous on I.

Method 1 for solving linear equations

(a) Write the equation in the standart form (2).

(b) Find the integrating factor $\mu(x)$ solving differential equation

$$\frac{d\mu}{dx} - P(x)\mu = 0.$$

(c) Integrate the equation

$$\frac{d}{dx}[\mu y] = \mu Q(x)$$

and solve for y by dividing by $\mu(x)$.

Method 2 (variation of parameter) for solving linear equations

(a) Write the equation in the standart form (2).

(b) Write the corresponding homogeneous equation

$$y' + P(x)y = 0, \tag{3}$$

which is obtained from (2) by replacing $Q(x)$ with zero.

(c) Find the solution to the homogeneous equation (3)

$$y_{\text{hom}}(x) = C\exp\left[-\int P(x)dx\right], \tag{4}$$

(d) Write the solution to the nonhomogeneous equation

$$y(x) = C(x)\exp\left[-\int P(x)dx\right], \tag{5}$$

where $C(x)$ is an unknown function.

(e) Find $C(x)$ integrating the equation

$$C'(x) = Q(x)\exp\left[\int P(x)dx\right].$$

(f) Substitute $C(x)$ into (5).

Existence and uniqueness of solution

Theorem 1. Suppose $P(x)$ and $Q(x)$ are continuous on some interval I that contains the point x_0. Then for any choice of initial value y_0, there exists a unique solution $y(x)$ on I to the initial value problem

$$y' + P(x)y = Q(x), \qquad y(x_0) = y_0. \tag{6}$$

Example 1. Obtain the general solution to the equation

$$y'\cos x + y\sin x = \cos^4 x.$$

SOLUTION We will assume, that $\cos x \neq 0$ or $x \neq \frac{\pi}{2} + 2\pi n$, $n = 0, \pm 1, \pm 2, \ldots$ so, we can divide an equation by $\cos x$

$$y' + y\frac{\sin x}{\cos x} = \cos^3 x,$$

For this equation $P(x) = \frac{\sin x}{\cos x}$, $Q(x) = \cos^3 x$.
I will solve this equation using method 1. So, equation for μ is

$$\mu' - \frac{\sin x}{\cos x}\mu = 0$$

.

Seprating the variables and integrating gives

$$\frac{d\mu}{\mu} = \frac{\sin x}{\cos x}dx,$$

$$\int \frac{d\mu}{\mu} = \int \frac{\sin x}{\cos x}dx,$$

$$\ln|\mu(x)| = -\ln|\cos x|,$$

$$\mu(x) = \frac{1}{\cos x}.$$

Lets find y

$$\frac{d}{dx}\left[\frac{1}{\cos x}y\right] = \frac{1}{\cos x}\cos^3 x,$$

$$\frac{1}{\cos x}y = \int \cos^2 x dx = \int \frac{\cos 2x + 1}{2}dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C,$$

so,

$$y(x) = \frac{1}{2}x\cos x + \frac{1}{4}\sin 2x\cos x + C\cos x.$$

Example 2. Solve the initial value problem.

$$dy - ydx - 2x\mathrm{e}^x dx = 0, \quad y(0) = \mathrm{e} - 2$$

Lets rewrite this equation in the form

$$y' - y = 2x\mathrm{e}^x.$$

I'm going to solve this equation using method 2. The corresponding homogeneous equation is

$$y' - y = 0$$

and the solution to this equation is

$$\ln|y| = x + C,$$

or

$$y = C_1\mathrm{e}^x,$$

where $C_1 = \mathrm{e}^C$.
Then the solution to nonhomogeneous equation is

$$y = C_1(x)\mathrm{e}^x,$$

here $C_1(x)$ is an unknown function.
Lets find y'.

$$y' = C_1'(x)\mathrm{e}^x + C_1(x)\mathrm{e}^x.$$

Substituting expression for y and y' into nonhomogeneous equation gives

$$C_1'(x)\mathrm{e}^x + C_1(x)\mathrm{e}^x - C_1(x)\mathrm{e}^x = 2x\mathrm{e}^x,$$

$$C_1'(x)\mathrm{e}^x = 2x\mathrm{e}^x,$$

$$C_1'(x) = 2x,$$

$$C_1(x) = x^2 + C_2.$$

Thus, the solution to the nonhomogeneous equation is

$$y = (x^2 + C_2)\mathrm{e}^x.$$