Chapter 4. Linear Second Order Equations

Chapter 4.1 is skipped.

Section 4.2 **Linear Differential Operators**

A **linear second order equation** is an equation that can be written in the form

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = b(x). \tag{1}$$

We will assume that $a_0(x)$, $a_1(x)$, $a_2(x)$, $b(x)$ are continuous functions of x on an interval I. When a_0, a_1, a_2, b are constants, we say the equation has **constant coefficients**, otherwise it has **variable coefficients**.

For now, we are interested in those linear equations for which $a_2(x)$ is never zero on I. In that case we can rewrite (1) in the **standard form**

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = g(x), \tag{2}$$

where $p(x) = a_1(x)/a_2(x)$, $q(x) = a_0(x)/a_2(x)$ and $g(x) = b(x)/a_2(x)$ are continuous on I.

Associated with equation (2) is the equation

$$y'' + p(x)y' + q(x)y = 0, \tag{3}$$

which is obtained from (2) by replacing $g(x)$ with zero. We say that equation (2) is a **nonhomogeneous equation** and that (3) is the corresponding **homogeneous equation**.

Let's consider the expression on the left-hand side of equation (3),

$$y''(x) + p(x)y'(x) + q(x)y(x). \tag{4}$$

Given any function y with a continuous second derivative on the interval I, then (4) generates a new function

$$L[y] = y''(x) + p(x)y'(x) + q(x)y(x). \tag{5}$$

What we have done is to associate with each function y the function $L[y]$. This function L is defined *on a set of functions*. Its domain is the collection of functions with continuous second derivatives; its range consists of continuous functions; and the rule of correspondence is given by (5). We will call this mappings **operators**. Because L involves differentiation, we refer to L as a **differential operator**.

The image of a function t under the operator L is the function $L[y]$. If we want to evaluate this image function at some point x, we write $L[y](x)$.

Example 1. Let $L[y](x) = x^2y''(x) - 3xy'(x) - 5y(x)$. Compute
(a) $L[\cos x]$; (b) $L[x^{-1}]$;
(c) $L[e^{rx}]$, r a constant.

SOLUTION.
(a) If $y = \cos x$, then $y' = -\sin x$ and $y'' = -\cos x$, so

$$L[y](x) = x^2y''(x) - 3xy'(x) - 5y(x) = x^2(-\cos x) - 3x(-\sin x) - 5\cos x = -(x^2+5)\cos x + 3x\sin x.$$

Thus, L maps the function $\cos x$ to the function $-(x^2+5)\cos x + 3x\sin x$.

(b) If $y = x^{-1}$, then we similarly find

$$L[y](x) = x^2y''(x) - 3xy'(x) - 5y(x) = x^2\frac{2}{x^3} - 3x\left(-\frac{1}{x^2}\right) - 5\frac{1}{x} = \frac{2}{x} + \frac{3}{x} - \frac{5}{x} = 0.$$

Thus, L maps the function x^{-1} to the zero function (or $y = x^{-1}$ is the solution to the equation $x^2y''(x) - 3xy'(x) - 5y(x) = 0$).

(c) If $y = \mathrm{e}^{rx}$,

$$L[y](x) = x^2y''(x) - 3xy'(x) - 5y(x) = x^2r^2\mathrm{e}^{rx} - 3xr\mathrm{e}^{rx} - 5\mathrm{e}^{rx} = ((xr)^2 - 3xr - 5)\mathrm{e}^{rx}.$$

Thus, L maps the function e^{rx} to the function $((xr)^2 - 3xr - 5)\mathrm{e}^{rx}$.

The differential operator L defined by (5) has two very important properties.

Lemma 1. Let $L[x] = y''(x) + p(x)y'(x) + q(x)y(x)$. If y, y_1, and y_2 are any twice-differentiable functions on the interval I and if c is any constant, then

$$L[y_1 + y_2] = L[y_1] + L[y_2], \tag{6}$$

$$L[cy] = cL[y]. \tag{7}$$

Proof. On I, we have

$$L[y_1+y_2] = (y_1+y_2)'' + p(x)(y_1+y_2)' + q(x)(y_1+y_2) = (y_1''+y_2'') + p(x)(y_1'+y_2') + q(x)(y_1+y_2) =$$

$$(y_1'' + p(x)y_1' + q(x)y_1) + (y_2'' + p(x)y_2' + q(x)y_2) = L[y_1] + L[y_2],$$

which verifies property (6).
Similarly we can prove property (7).

Any operator that satisfied satisfies properties (6) and (7) for any constant c and any functions y, y_1, and y_2 in its domain is called a **linear operator** and we can say that *"L preserves linear combination"*. If (6) or (7) fails to hold, the operator is **nonlinear**.

Lemma 1 says that the operator L, defined by (5) is linear.

Example 2. Show that T defined by

$$T[y] = y'' + \{y'y^2\}^{1/3}$$

is a nonlinear.

SOLUTION. To demonstrate that T is nonlinear, it suffices to show that property (6) (or (7)) is *not always* satisfied. Let's try $y_1(x) = x^2$ and $y_2(x) = x^3$. Since $y_1'(x) = 2x$, $y_1'(x) = 2$, $y_2'(x) = 3x^2$, $y_2''(x) = 6x$,

$$T[y_1] = 2 + \{2x \cdot x^4\}^{1/3} = 2 + \{2x^5\}^{1/3},$$

$$T[y_2] = 6x + \{3x^2 \cdot x^6\}^{1/3} = 6x + \{3x^8\}^{1/3},$$

$$T[y_1 + y_2] = 2 + 6x + \{(2x^5 + 3x^8)(x^2 + x^3)^2\}^{1/3} \neq 2 + \{2x^5\}^{1/3} + 6x + \{3x^7\}^{1/3},$$

so $T[y_1] + T[y_2] \neq T[y_1 + y_2]$, property (6) is violated. Hence, T is a nonlinear operator.

The linearity of the differential operator L in (5) can be used to prove the following theorem concerning homogeneous equations.

Theorem 1 (linear combination of solutions). Let y_1 and y_2 be solutions to the *homogeneous* equation (3). Then any linear combination $C_1y_1 + C_2y_2$ of y_1 and y_2, where C_1 and C_2 are constants, is also the solution to (3).

Proof. Since y_1 and y_2 are solutions to (3), then $L[y_1] = 0$ and $L[y_2] = 0$. Using the linearity of L, we have

$$L[C_1y_1 + C_2y_2] = L[C_1y_1] + L[C_2y_2] = C_1L[y_1] + C_2L[y_2] = 0$$

for any C_1 and C_2. Thus, $C_1y_1 + C_2y_2$ is a solution to (3).

Example 3. Given that $y_1(x) = \mathrm{e}^{2x}\cos x$ and $y_2(x) = \mathrm{e}^{2x}\sin x$ are solutions to the homogeneous equation

$$y'' - 4y' + 5y = 0,$$

find solutions to this equation that satisfy the following initial conditions:
(a) $y(0) = 2$, $y'(0) = 1$.
(b) $y(\pi) = 4\mathrm{e}^{2\pi}$, $y'(\pi) = 5\mathrm{e}^{2\pi}$.

SOLUTION. As a consequence of T.1, any linear combination

$$y(x) = C_1\mathrm{e}^{2x}\cos x + C_2\mathrm{e}^{2x}\sin x$$

with C_1 and C_2 arbitrary consrants, will be the solution also.

$$y'(x) = 2C_1\mathrm{e}^{2x}\cos x - C_1\mathrm{e}^{2x}\sin x + 2C_2\mathrm{e}^{2x}\sin x + C_2\mathrm{e}^{2x}\cos x =$$

$$(2C_1 + C_2)\mathrm{e}^{2x}\cos x + (2C_2 - C_1)\mathrm{e}^{2x}\sin x.$$

(a)

$$y(0) = C_1(1)(1) + C_2(1)(0) = C_1 = 2,$$

$$y'(0) = (2C_1 + C_2)(1) + (2C_2 - C_1)(0) = 2C_1 + C_2 = 1.$$

Since $C_2 = 1 - 2C_1$ and $C_1 = 2$, $C_2 = -3$. Hence, the solution to the given initial value problem is

$$y(x) = 2\mathrm{e}^{2x}\cos x - 3\mathrm{e}^{2x}\sin x.$$

(b)

$$y(\pi) = C_1 e^{2\pi}(-1) + C_2 e^{2\pi}(0) = -C_1 e^{2\pi} = 4e^{2\pi},$$

so, $C_1 = -4$.

$$y'(\pi) = (2C_1 + C_2)e^{2\pi}(-1) + (2C_2 - C_1)e^{2\pi}(0) = -(2C_1 + C_2)e^{2\pi} = 5e^{2\pi},$$

so, $2C_1 + C_2 = -5$. Since $C_2 = -5 - 2C_1$ and $C_1 = -4$, $C_2 = 3$. Hence, the solution to the given initial value problem is

$$y(x) = -4e^{2x}\cos x + 3e^{2x}\sin x.$$