MATH 308 Fall 2007 Practice Exam I (solutions)

2. Correct answer is (b)

3. Correct answer is (a)

4. Correct answer is (b)

5. Correct answer is (d)

6. Correct answer is (b)

7. Correct answer is (a)

8. Find a **general** solution to the equation

$$y'' + 6y' + 9y = \frac{e^{-3x}}{1+2x}$$

SOLUTION. The corresponding homogeneous equation is

$$y'' + 6y' + 9y = 0.$$

The fundamental solution set to this homogeneous equation is $\{\mathrm{e}^{-3x}, x\mathrm{e}^{-3x}\}$,

$$y_1(x) = \mathrm{e}^{-3x}, \quad y_2(x) = x\mathrm{e}^{-3x},$$
$$y_1'(x) = -3\mathrm{e}^{-3x}, \quad y_2'(x) = \mathrm{e}^{-3x} - 3x\mathrm{e}^{-3x} = (1-3x)\mathrm{e}^{-3x},$$

The general solution to the homogeneous equation is

$$y_h(x) = c_1\mathrm{e}^{-3x} + c_2 x\mathrm{e}^{-3x}.$$

Then the particular solution to the nonhomogeheous equation is

$$y_p(x) = c_1(x)\mathrm{e}^{-3x} + c_2(x)x\mathrm{e}^{-3x}.$$

To find $c_1(x)$ and $c_2(x)$ we have to solve the system

$$\begin{cases} c_1'(x)\mathrm{e}^{-3x} + c_2'(x)x\mathrm{e}^{-3x} = 0 \\ c_1'(x)(-3\mathrm{e}^{-3x}) + c_2'(x)(1-3x)\mathrm{e}^{-3x} = \frac{\mathrm{e}^{-3x}}{1+2x}. \end{cases}$$

Since $\mathrm{e}^{-3x} \neq 0$, we can divide both equations by e^{-3x}.

$$\begin{cases} c_1'(x) + c_2'(x)x = 0 \\ -3c_1'(x) + c_2'(x)(1-3x) = \frac{1}{1+2x}. \end{cases}$$

Substituting $c_1'(x) + c_2'(x)x = 0$ into second equation gives

$$c_2'(x) = \frac{1}{1+2x}.$$

Integrating gives

$$c_2(x) = \frac{1}{2}\ln|1+2x| + c_3.$$

Since $c_1'(x) = -c_2'(x)x = -\frac{x}{1+2x}$.
Integrating gives

$$c_1(x) = -\int \frac{x}{1+2x}dx = -\int \left(\frac{1}{2} - \frac{1}{2}\frac{1}{1+2x}\right)dx = -\frac{x}{2} + \frac{1}{4}\ln|1+2x| + c_4.$$

Thus, the general solution to the given nonhomogneous equation is

$$y(x) = \left[-\frac{x}{2} + \frac{1}{4}\ln|1+2x| + c_4 + x\left(\frac{1}{2}\ln|1+2x| + c_3\right)\right]\mathrm{e}^{-3x}.$$

9. Solve the following initial value problem:

$$y' = \frac{4x-3}{2y+6}, \quad y(1) = -5.$$

Write your solution in the **explicit** form.

SOLUTION. Separating variables and integrating gives

$$(2y+6)dy = (4x-3)dx,$$
$$\int (2y+6)dy = \int (4x-3)dx,$$
$$y^2 + 6y = 2x^2 - 3x + c.$$

Substituting $y(1) = -5$ gives

$$25 - 30 = 2 - 3 + c,$$
$$c = -4.$$

Thus, the **implicit** solution to the initial value problem is

$$y^2 + 6y = 2x^2 - 3x - 4.$$

The **explicit solution** is

$$y(x) = \frac{-6 \pm \sqrt{36 - 4(2x^2 - 3x + 4)}}{2}.$$

10. Find a **particular** solution to the equation

$$4y'' + y' = 4x^3 + 48x^2 + 1$$

SOLUTION. Let's find the general solution to the corresponding homogeneous equation

$$4y'' + y' = 0.$$

The associated auxiliary equation is

$$4r^2 + r = r(4r + 1) = 0,$$

which has two roots $r = 0$ and $r = -1/4$. Thus, the general solution to the homogeneous equation is

$$y_h(x) = c_1 + c_2 \mathrm{e}^{-x/4}.$$

Since $r = 0$ is one of two roots to the auxiliary equation and $m_1 = 3$, we seek a particular solution to the nonhomogeneous equation of the form

$$y_p(x) = x(Ax^3 + Bx^2 + Cx + D) = Ax^4 + Bx^3 + Cx^2 + Dx,$$

where A, B, C, and D are unknowns.
Now we have to substitute $y_p(x)$, $y_p'(x)$, and $y_p''(x)$ into equation.

$$y_p'(x) = 4Ax^3 + 3Bx^2 + 2Cx + D,$$

$$y_p''(x) = 12Ax^2 + 6Bx + 2C,$$

$$4y_p''(x) + y_p'(x) = 48Ax^2 + 24Bx + 8C + 4Ax^3 + 3Bx^2 + 2Cx + D =$$
$$= 4Ax^3 + (48A + 3B)x^2 + (24B + 2C)x + 8C + D = 4x^3 + 48x^2 + 1$$

Two polynomials are equal when corresponding coefficients are equal, so we set

$$\begin{aligned} x^3 &: \quad 4A = 4, \\ x^2 &: \quad 48A + 3B = 48, \\ x^1 &: \quad 24B + 2C = 0, \\ x^0 &: \quad 8C + D = 1 \end{aligned}$$

Solving the system gives $A = D = 1$, $B = C = 0$. So,

$$y_p(x) = x^4 + x.$$

11. The correct answer is

$$W[y_1, y_2] = \frac{x\mathrm{e}^x}{\mathrm{e}} = x\mathrm{e}^{x-1}$$

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12. The correct answer is $y(x) = \mathrm{e}^{-\frac{3}{2}x}\left(\cos x + \frac{1}{2}\sin x\right)$.

13. Find the general solution to

$$\frac{1}{x}\frac{dy}{dx} - \frac{2y}{x^2} = x\cos x.$$

SOLUTION. To put this equation in standard form, we multiply by x to obtain

$$y' - \frac{2y}{x} = x^2\cos x.$$

Here $P(x) = -\frac{2}{x}$, $Q(x) = x^2 \cos x$.
We can find the integrating factor $\mu(x)$ solving the equation

$$\mu' = -\frac{2}{x}\mu.$$

Separating the variables and integrating gives

$$\ln|\mu| = -2\ln|x|,$$
$$\mu(x) = \frac{1}{x^2}.$$

Now, we can find y from the equation

$$\frac{d}{dx}\left[\frac{1}{x^2}y\right] = \frac{1}{x^2}x^2 \cos x$$

Integrating gives

$$\frac{y}{x^2} = \int \cos x dx = \sin x + c.$$

Thus, the general solution to the given equation is

$$y(x) = x^2(\sin x + c).$$

14. The correct answer is

$$y(x) = c_1 x + \frac{c_2}{x}.$$