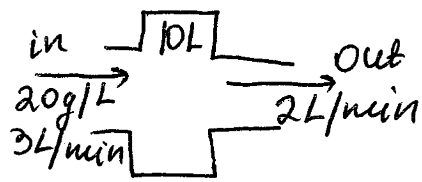
1. A large tank initially contains 10 L of fresh water. A brine containing 20 g/L of salt flows into the tank at a rate of 3 L/min. The solution inside the tank is kept well stirred and flows out of the tank at the rate 2 L/min. Determine the concentration of salt in the tank as a function of time.

Let $x(t)$ be the mass of salt in the tank at time t.



$$\frac{dx}{dt} = \text{INPUT RATE} - \text{OUTPUT RATE}$$

$$\text{INPUT RATE} = 20\,(g/L)\cdot 3\,(L/min) = 60\ g/min$$

$$\text{OUTPUT RATE} = \frac{x(t)\cdot 2\,(L/min)}{10+(3-2)t} = \frac{2x(t)}{10+t}\ (g/min)$$

$$\frac{dx}{dt} = 60 - \frac{2}{10+t}x$$

$$x(0)=0.$$

Solve the equation:

$$\frac{dx}{dt} + \frac{2}{10+t}x = 60 - \text{a linear equation.}$$

Integrating factor μ: $\frac{d\mu}{dt} - \frac{2}{10+t}\mu = 0$

$$\frac{d\mu}{\mu} = \frac{2}{10+t}dt$$

$$\mu = (10+t)^2$$

$$\frac{d}{dt}\left((10+t)^2 x\right) = (10+t)^2 60$$

$$(10+t)^2 x = 60\frac{(10+t)^3}{3} + C$$

$$x(t) = 20(10+t) + \frac{C}{(10+t)^2}\ ;\ x(0) = 200 + \frac{C}{100} = 0$$

$$C = -20000$$

$$x(t) = 20(10+t) - \frac{20000}{(10+t)^2}.$$

concentration of salt $= \boxed{\frac{x(t)}{10+t} = 20 - \frac{20000}{(10+t)^3}}$

2. Suppose that a sum S_0 is invested at an annual rate of return r compounded continuously.

(a) Find the time T required for the original sum to double in value as a function of r.

$\frac{dS(t)}{dt} = rS(t)$. - separable equation

$S(0) = S_0$.

$\frac{dS}{S} = r\,dt$

$\ln|S| = rt + C$

$S = Ce^{rt}$

$S(0) = C = S_0$

$S(t) = S_0 e^{rt}$.

Find T such that $S(T) = 2S_0$:

$S(T) = S_0 e^{rT} = 2S_0$

$e^{rT} = 2$

$\boxed{T = \frac{\ln 2}{r}}$

(b) Determine T if $r = 7\%$.

$T = \frac{\ln 2}{0.07} \approx \boxed{9.9 \text{ years}}$

(c) Find the return rate that must be achieved if the initial investment is to double in 8 years.

$T = \frac{\ln 2}{r} \longrightarrow r = \frac{\ln 2}{T} = \frac{\ln 2}{8} \approx 0.0866$

$\boxed{r = 8.66\%}$

3. An object with temperature 150^0 is placed in a freezer whose temperature is 30^0. Assume that the temperature of the freezer remains essentially constant.

(a) If the object is cooled to 120^0 after 8 min, what will its temperature be after 18 min?

$T(t)$ – temperature of the object at time t.

$\frac{dT}{dt} = k(T-30)$ – separable equation

$T(0)=150$

$\frac{dT}{T-30} = k\,dt$

$\ln|T-30| = kt + C$

$T(t) = 30 + Ce^{kt}$

$T(0) = 30 + C = 150 \to C = 120$

$T(t) = 30 + 120e^{kt}$

$T(t) = 30 + 120e^{-0.36t}$

$T(18) = 30 + 120e^{-0.36\cdot 18}$

$=$

$T(8)=120$

k – ?

$T(8) = 30 + 120e^{8k}$

$30 + 120e^{8k} = 120$

$120e^{8k} = 90$

$e^{8k} = \frac{3}{4}$

$8k = \ln\frac{3}{4}$

$k = \frac{1}{8}\ln\frac{3}{4}$

$\approx -0.36.$

(b) When will its temperature be 60^0?

$T=60$; t – ?

$60 = 30 + 120e^{-0.36t}$

$30 = 120e^{-0.36t}$

$t = -\frac{1}{0.36}\ln\frac{1}{4}$

$\approx \boxed{3.85(\text{min})}$

4. Determine (without solving the problem) an interval in which the solution to the initial value problem

$$(4-t^2)y' + 2ty = 3t^2, \quad y(1) = -3$$

is certain to exist.

$y' + \frac{2t}{4-t^2}y = \frac{3t^2}{4-t^2}$

$y(1) = -3$

Functions $P(t) = \frac{2t}{4-t^2}$ and $Q(t) = \frac{3t^2}{4-t^2}$

continuous on $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

$-2 \quad 2 \quad t$

Thus, the solution to the IVP is certain to exist on $\boxed{(-2, 2)}$.

5. Solve the initial value problem

$$y' = \frac{t^2}{1+t^3}, \quad y(0) = y_0$$

and determine how the interval in which the solution exists depends on the initial value y_0.

solve the initial value problem.

$$\frac{dy}{dt} = \frac{t^2}{1+t^3}$$

$$dy = \frac{t^2}{1+t^3}\, dt$$

$$y = \frac{1}{3} \ln|1+t^3| + C$$

$$y(0) = C = y_0$$

$$y = \frac{1}{3} \ln|1+t^3| + y_0$$

The interval on which solution exists does not depend on y_0.

6. Solve the following initial value problem

$$\sqrt{y}dt + (1+t)dy = 0 \quad y(0) = 1.$$

Separate variables:

$$\sqrt{y}\, dt = -(1+t)\, dy$$

$$\frac{dy}{\sqrt{y}} = \cancel{-(1+t)d} \; -\frac{dt}{1+t}$$

$$\frac{y^{-1/2+1}}{-1/2+1} = -\ln|1+t| + C$$

$$2\sqrt{y} = -\ln|1+t| + C$$

$$\sqrt{y} = \frac{1}{2}(C - \ln|1+t|)$$

$$y = \frac{1}{4}(C - \ln|1+t|)^2$$

$$1 = y(0) = \frac{1}{4} C^2 \longrightarrow C^2 = 4$$

$$C = 2$$

$$\boxed{y = \frac{1}{4}(2 - \ln|1+t|)^2}$$

7. Find the general solution to the equation

$$(t^2 - 1)y' + 2ty + 3 = 0$$

$$y' + \frac{2t}{t^2-1} y = -\frac{3}{t^2-1}$$ – linear equation

Integrating factor

$$\mu: \quad \frac{d\mu}{dt} - \frac{2t}{t^2-1}\mu = 0$$

$$\frac{d\mu}{dt} = \frac{2t}{t^2-1}\mu$$

$$\frac{d\mu}{\mu} = \frac{2t}{t^2-1}\, dt$$

$$\ln|\mu| = \ln|t^2-1|$$

$$\mu = t^2-1$$

$$\frac{d}{dt}\left((t^2-1)y\right) = -\frac{3}{t^2-1}(t^2-1)$$

$$(t^2-1)y = -3t + C$$

$$\boxed{y = -\frac{3t}{t^2-1} + \frac{C}{t^2-1}}$$

8. Solve the initial value problem

$$(ye^{xy}\cos(2x) - 2e^{xy}\sin(2x) + 2x)dx + (xe^{xy}\cos(2x) - 3)dy = 0, \quad y(0) = -1$$

$M(x,y) = ye^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x$

$N(x,y) = xe^{xy}\cos 2x - 3$

$\frac{\partial M}{\partial y} = e^{xy}\cos 2x + xye^{xy}\cos 2x - 2xe^{xy}\sin 2x$

$\frac{\partial N}{\partial x} = e^{xy}\cos 2x + xye^{xy}\cos 2x - 2xe^{xy}\sin 2x$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \longrightarrow$ EXACT EQUATION.

$\varphi(x,y)$ such that $\frac{\partial \varphi}{\partial x} = M$, $\frac{\partial \varphi}{\partial y} = N$:

$$\begin{cases} \frac{\partial \varphi}{\partial x} = ye^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x \\ \frac{\partial \varphi}{\partial y} = xe^{xy}\cos 2x - 3 \end{cases}$$

integrate the 2nd equation with respect to y:

$\varphi(x,y) = x\frac{1}{x}e^{xy}\cos 2x - 3y + g(x)$

$= e^{xy}\cos 2x - 3y + g(x)$

Plug $\varphi(x,y)$ into the 1st equation:

$\frac{\partial \varphi}{\partial x} = ye^{xy}\cos 2x + e^{xy}\sin 2x(-2) + g'(x)$

$= \cancel{ye^{xy}\cos 2x + xye^{xy}}$

$= ye^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x$

$g'(x) = 2x$, $g(x) = x^2 + C$

$\varphi(x,y) = e^{xy}\cos 2x - 3y + x^2 + C$

The general solution to the equation is

$$e^{xy}\cos 2x - 3y + x^2 + C = 0$$

plug it into the initial condition

$$y(0) = -1. \quad (x_0 = 0,\ y_0 = -1)$$

$$e^{0(-1)}\cos(0) - 3(-1) + (0)^2 + C = 0$$

$$1 + 3 + C = 0$$

$$C = -4$$

The solution to the initial value problem is

$$\boxed{e^{xy}\cos 2x - 3y + x^2 - 4 = 0.}$$