Math 308 Fall 2007 Review Before Final.

Chapter I **Introduction**

Section 1.1 **Background**

Definition Equation that contains some derivatives of an unknown function is called a *differential equation*.

Definition A differential equation involving only ordinary derivatives with respect to a single variable is called an *ordinary differential equations* or ODE. A differential equation involving partial derivatives with respect to more then one variable is a *partial differential equations* or PDE.

Definition The *order* of a differential equation is the order of the highest-order derivatives present in equation.

Definition An ODE is *linear* if it has format

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + a_1(x)\frac{dy}{dx} + a_0(x)y = F(x),$$

where $a_n(x)$, $a_{n-1}(x)$,...,$a_0(x)$ and $F(x)$ depend only on variable x. If an ODE is not linear, then we call it **nonlinear**.

Section 1.2 **Solutions and Initial Value Problems**

A general form for an nth-order equation with variable x and unknown function $y = y(x)$ can be expressed as

$$F\left(x, y, \frac{dy}{dx}, \cdots, \frac{d^n y}{dx^n}\right) = 0,$$

where F is a function that depends on x, y, and the derivatives of y up to the order n. We assume that the equation holds for all x in an open interval I ($a < x < b$, where a or b could be infinite). In many cases we can isolate the highest-order term $d^n y/dx^n$ and write equation (1) as

$$\frac{d^n y}{dx^n} = f\left(x, y, \frac{dy}{dx}, \cdots, \frac{d^{n-1}y}{dx^{n-1}}\right) = 0.$$

Definition. A function $\varphi(x)$ that when substituted by y in equation (1) or (2) satisfies the equation for all x in the interval I is called an *explicit solution* to the equation on I.

Definition. A relation $G(x, y) = 0$ is said to be an *implicit solution* to equation (1) on the interval I if it defines one or more explicit solutions on I.

Definition. By an *initial value problem* for an nth-order differential equation

$$F\left(x, y, \frac{dy}{dx}, \cdots, \frac{d^n y}{dx^n}\right) = 0$$

we mean: Find a solution to the differential equation on an interval I that satisfies at x_0 the n initial conditions:

$$\begin{aligned} y(x_0) &= y_0, \\ \frac{dy}{dx}(x_0) &= y_1, \\ &\vdots \\ \frac{d^{n-1}y}{dx^{n-1}}(x_0) &= y_{n-1}, \end{aligned}$$

where $x_0 \in I$ and y_0, y_1,...,y_{n-1} are given constants.

In case of a first-order equation $F\left(x, y, \frac{dy}{dx}\right)$, the initial conditions reduce to the single requirement

$$y(x_0) = y_0,$$

and in the case of a second-order equation, the initial conditions have the form

$$y(x_0) = y_0, \qquad \frac{dy}{dx} = y_1.$$

Theorem. Given the initial value problem

$$\frac{dy}{dx} = f(x, y), \qquad y(0) = y_0,$$

assume that f and $\partial f / \partial y$ are continuous functions in a rectangle

$$R = \{(x, y) : a < x < b, c < y < d\}$$

that contains the point (x_0, y_0). Then the initial value problem has a unique solution $\varphi(x)$ in some interval $x_0 - \delta < x < x_0 + \delta$, where δ is a positive number.

Chapter 2. First-Order Differential Equations

Section 2.2 **Separable Equations**

$$\frac{dy}{dx} = f(x, y)$$

Sometimes function $f(x, y)$ can be represented as a product of two functions, one of which depends ONLY on x, another depends ONLY on y, or $f(x, y) = g(x)h(y)$. Then

$$\frac{dy}{dx} = g(x)h(y).$$

$$\frac{dy}{h(y)} = g(x)dx,$$

$$\int \frac{dy}{h(y)} = \int g(x)dx,$$

Thus, the solution to an equation is

$$H(y) = G(x) + C,$$

here $H(y)$ is an antiderivative of $1/h(y)$, $G(x)$ is an antiderivative of $g(x)$, C is a constant.

Section 2.3 **Linear Equations**

A **linear first-order equation** is an equation that can be expressed in the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = b(x),$$

where $a_0(x)$, $a_1(x)$, $b(x)$ depend only on x.

We will assume that $a_0(x)$, $a_1(x)$, $b(x)$ are continuous functions of x on an interval I. We are interested in those linear equations for which $a_1(x)$ is never zero on I. In that case we can rewrite linear equation in the **standard form**

$$\frac{dy}{dx} + P(x)y = Q(x),$$

where $P(x) = a_0(x)/a_1(x)$ and $Q(x) = b(x)/a_1(x)$ are continuous on I.

There are two methods of solving linear first-order differential equations.

Method 1 for solving linear equations

(a) Write the equation in the standard form $\frac{dy}{dx} + P(x)y = Q(x)$.

(b) Find the integrating factor $\mu(x)$ solving differential equation

$$\frac{d\mu}{dx} - P(x)\mu = 0.$$

(c) Integrate the equation

$$\frac{d}{dx}[\mu y] = \mu Q(x)$$

and solve for y by dividing by $\mu(x)$.

Method 2 (variation of parameter) for solving linear equations

(a) Write the equation in the standard form $\frac{dy}{dx} + P(x)y = Q(x)$.

(b) Write the corresponding homogeneous equation

$$y' + P(x)y = 0,$$

which is obtained from $\frac{dy}{dx} + P(x)y = Q(x)$ by replacing $Q(x)$ with zero.

(c) Find the solution to the homogeneous equation

$$y_{\text{hom}}(x) = C \exp\left[-\int P(x)dx\right],$$

(d) Write the solution to the nonhomogeneous equation

$$y(x) = C(x) \exp\left[-\int P(x)dx\right],$$

where $C(x)$ is an unknown function.
(e) Find $C(x)$ integrating the equation

$$C'(x) = Q(x) \exp\left[\int P(x)dx\right].$$

(f) Substitute $C(x)$ into $y(x) = C(x) \exp\left[-\int P(x)dx\right]$.

Existence and uniqueness of solution

Theorem. Suppose $P(x)$ and $Q(x)$ are continuous on some interval I that contains the point x_0. Then for any choice of initial value y_0, there exists a unique solution $y(x)$ on I to the initial value problem

$$y' + P(x)y = Q(x), \qquad y(x_0) = y_0.$$

Chapter 3. Mathematical models and numerical methods involving first-order equations

Section 3.2 **Compartmental analysis**

Mixing problem.

A brine solution of salt flows at a constant rate of a L/min into a large tank that initially held A L of brine solution in which was dissolved b kg of salt. The solution inside the tank is kept well stirred and flows out of the tank at the rate c L/min. If the concentration of salt in the brine entering the tank is d kg/L, determine the mass of salt in the tank after t min.

SOLUTION. Let $x(t)$ denote the mass of salt in the tank at time t, we can determine the concentration of salt in the tank by dividing $x(t)$ by the volume of fluid in the tank at time t.

First, we must determine the rate at which salt enters the tank. We are given that brine flows into the tank at a rate of a L/min. Since the concentration is b kg/L, we conclude that the input rate of salt into the tank is

$$ab \text{ kg/min}$$

We must now determine the output rate of salt from the tank. The brine solution in the tank is well stirred, so let's assume that the concentration of salt in the tank is uniform. So, the concentration of salt in any part of the tank at time t is just $x(t)$ divided by the volume of fluid in the tank. The output rate is

$$\frac{cx(t)}{A + (c - a)t} \text{ kg/min}$$

The tank initially contains d kg of salt, so we set $x(0) = d$. The initial value problem

$$\frac{dx}{dt} = \text{input rate} - \text{output rate} = ab - \frac{cx(t)}{A + (c-a)t}, \quad x(0) = d,$$

is a mathematical model for the mixing problem.

Population models

If we assume that the people die only of natural causes, we might expect the death rate also to be proportional to the size of the population. So,

$$\frac{dp}{dt} = k_1 p - k_2 p = (k_1 - k_2)p = kp,$$

where $k = k_1 - k_2$ and k_2 is the proportionality constant for the death rate. Let's assume that $k_1 > k_2$ so that $k > 0$. This gives the mathematical model

$$\frac{dp}{dt} = kp, \quad p(0) = p_0,$$

which is called the **Malthusian** or **exponential, law** of population growth.

The solution to this initial value problem is

$$p(t) = p_0 e^{kt}.$$

What about premature death? We might assume that another component of the death is proportional to the number of two-party interactions. There are $p(p-1)/2$ such possible interactions for a population of size p. Thus, if we combine the birth rate with the death rate and rearrange constants, we get the **logistic model**

$$\frac{dp}{dt} = -Ap(p - p_1), \quad p(0) = p_0,$$

where $A = k_3/2$ and $p_1 = (2k_1/k_3) + 1$.

The function $p(t)$ is called the **logistic function**.

Section 3.3 **Heating and cooling of buildings.**

Let $T(t)$ represent the temperature inside the building at time t and view the building as a single compartment.

We will consider three main factors that affect the temperature inside the building. First is the heat produced by people, lights, and machines inside the building. This causes a rate of increase in temperature that we will denote by $H(t)$. Second is the heating (or cooling) supplied by the furnace (or air conditioner). This rate of increase (or decrease) in temperature will be represented by $U(t)$. The third factor is the effect of the outside temperature $M(t)$ on the temperature inside the building. Third factor can be modeled using **Newtons law of cooling**. This law states that

$$\frac{dT}{dt} = K(M(t) - T(t)).$$

The positive constant K depends on the physical properties of the building, K does not depend on M, T or t.

Summarizing, we have

$$\frac{dT}{dt} = K(M(t) - T(t)) + U(t) + H(t),$$

where $H(t) \geq 0$ and $U(t) > 0$ for furnace heating and $U(t) < 0$ for air conditioning cooling.

Section 3.4 **Newtonian mechanics**

Procedure for Newtonian models

(a) Determine all relevant forces acting on the object being studied. It is helpful to draw a simple diagram of the object that depicts these forces.

(b) Choose an appropriate axis or coordinate system in which to represent the motion of the object and the forces acting on it.

(c) Apply Newton's second law to determine the equations of motion for the object.

You have to remember that the gravitational acceleration is approximately equal in the U.S. Customary System $g = 32$ ft/sec^2, and in the meter-kilogram-second system $g = 9.81$ m/sec^2.

An object of mass m is given an initial downward velocity v_0 and allowed to fall under the influence of gravity. Assuming the gravitational force is constant and the force due to air resistance is proportional to the velocity of the object, determine the equation of motion for this body.

SOLUTION. Two forces are acting on the object: a constant force due to downward pull of gravity and force due to air resistance that is proportional to the velocity of the object and acts in opposition to the motion of the object. Hence, the motion of the object will take place along a vertical axis. On this axis we choose the origin to be the point where the object was initially dropped and let $x(t)$ denote the distance the object has fallen in time t.

The force due to gravity is

$$F_1 = mg,$$

where g is the acceleration due to gravity an Earth's surface. The force of air resistance is

$$F_2 = -bv(t),$$

where $b > 0$ is the proportionality constant. The net force acting on the object is

$$F = F_1 + F_2 = mg - bv(t).$$

We now apply Newton's second law:

$$m\frac{dv}{dt} = mg - bv(t).$$

Since, the initial velocity of the object is v_0, a model for the velocity of the falling body is expressed by the initial value problem

$$m\frac{dv}{dt} = mg - bv(t), \qquad v(0) = v_0,$$

where g and b are positive constant.

Hence, the equation of the motion is

$$x(t) = \frac{mg}{b}t + \frac{m}{b}\left(v_0 - \frac{mg}{b}\right)(1 - e^{-\frac{b}{m}t}).$$

Chapter 4. Linear Second Order Equations

Section 4.2 **Linear Differential Operators**

A **linear second order equation** is an equation that can be written in the form

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = b(x).$$

We will assume that $a_0(x)$, $a_1(x)$, $a_2(x)$, $b(x)$ are continuous functions of x on an interval I. When a_0, a_1, a_2, b are constants, we say the equation has **constant coefficients**, otherwise it has **variable coefficients**. We are interested in those linear equations for which $a_2(x)$ is never zero on I. In that case we can rewrite linear second-order equation in the **standard form**

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = g(x), \tag{1}$$

where $p(x) = a_1(x)/a_2(x)$, $q(x) = a_0(x)/a_2(x)$ and $g(x) = b(x)/a_2(x)$ are continuous on I.

Associated with equation (1) is the equation

$$y'' + p(x)y' + q(x)y = 0, \tag{2}$$

which is obtained from (1) by replacing $g(x)$ with zero. We say that equation (1) is a **nonhomogeneous equation** and that (2) is the corresponding **homogeneous equation**.

Given any function y with a continuous second derivative on the interval I, then $y''+p(x)y'+q(x)y$ generates a new function

$$L[y] = y''(x) + p(x)y'(x) + q(x)y(x).$$

What we have done is to associate with each function y the function $L[y]$. This function L is defined *on a set of functions*. Its domain is the collection of functions with continuous second derivatives. We will call these mappings **operators**. Because L involves differentiation, we refer to L as a **differential operator**.

The image of a function t under the operator L is the function $L[y]$. If we want to evaluate this image function at some point x, we write $L[y](x)$.

Lemma Let $L[x] = y''(x)+p(x)y'(x)+q(x)y(x)$. If y, y_1, and y_2 are any twice-differentiable functions on the interval I and if c is any constant, then

$$L[y_1 + y_2] = L[y_1] + L[y_2],$$
$$L[cy] = cL[y].$$

Any operator that satisfied satisfies both properties from Lemma for any constant c and any functions y, y_1, and y_2 in its domain is called a **linear operator** and we can say that *"L preserves linear combination"*. If properties fails to hold, the operator is **nonlinear**.

Lemma says that the operator $L = y''(x) + p(x)y'(x) + q(x)y(x)$ is linear.

Theorem (linear combination of solutions). Let y_1 and y_2 be solutions to the *homogeneous* equation. Then any linear combination $C_1y_1 + C_2y_2$ of y_1 and y_2, where C_1 and C_2 are constants, is also the solution to (2).

Theorem (existence and uniqueness of solution). Suppose $p(x)$, $q(x)$, and $g(x)$ are continuous on some interval (a, b) that contains the point x_0. Then, for any choice of initial values y_0, y_1 there exists a unique solution $y(x)$ on the whole interval (a, b) to the initial value problem

$$y'' + p(x)y' + q(x)y = g(x),$$
$$y(x_0) = y_0, \quad y'(0) = y_1.$$

Section 4.3. **Fundamental solutions of homogeneous equations**

Theorem. Let y_1 and y_2 denote two solutions on I to

$$y'' + p(x)y' + q(x)y = 0,$$

Suppose at some point $x_0 \in I$ these solutions satisfy

$$y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0) \neq 0.$$

Then every solution to (2) on I can be expressed in the form

$$y(x) = C_1 y_1(x) + C_2 y_2(x),$$

where C_1 and C_2 are constants.

Definition. For any two differentiable functions y_1 and y_2, the determinant

$$W[y_1, y_2](x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_1'(x)y_2(x)$$

is called the **Wronskian** of y_1 and y_2.

Definition. A pair of solutions $\{y_1, y_2\}$ to $y'' + p(x)y' + q(x)y = 0$ on I is called **fundamental solution set** if

$$W[y_1, y_2](x_0) \neq 0$$

at some $x_0 \in I$.

Procedure for solving homogeneous equations
To determine all solutions to $y'' + p(x)y' + q(x)y = 0$:
(a) Find two solutions y_1 and y_2 that constitute a fundamental solution set.
(b) Form the linear combination

$$y(x) = C_1 y_1(x) + C_2 y_2(x),$$

to obtain the general solution.

Definition. Two functions y_1 and y_2 are said to be **linearly dependent on I** if there exist constants C_1 and C_2, not both zero, such that

$$C_1 y_1(x) + C_2 y_2(x) = 0$$

for all $x \in I$. If two functions are not linearly dependent, they are said to be **linearly independent**.

Theorem. Let y_1 and y_2 be solutions to the equation $y'' + p(x)y' + q(x)y = 0$ on I, and let $x_0 \in I$. Then y_1 and y_2 are linearly dependent on I if and only if the constant vectors

$$\begin{pmatrix} y_1(x_0) \\ y_1'(x_0) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} y_2(x_0) \\ y_2'(x_0) \end{pmatrix}$$

are linearly dependent.

Corollary. If y_1 and y_2 are solutions to $y'' + p(x)y' + q(x)y = 0$ on I, then the following statements are equivalent:

(i) $\{y_1, y_2\}$ is a fundamental solution set on I.

(ii) y_1 and y_2 are linearly independent on I.

(iii) $W[y_1, y_2]$ is never zero on I.

Another representation of the Wronskian for two solutions $y_1(x)$ and $y_2(x)$ to the equation $y'' + py' + qy = 0$ on (a, b) is **Abel's identity:**

$$W[y_1, y_2](x) = C \exp\left[-\int_{x_0}^{x} p(t)dt\right],$$

where $x_0 \in (a, b)$ and C is a constant that depends on y_1 and y_2.

Section 4.4 **Reduction of Order**

A general solution to a linear second order homogeneous equation is given by a linear combination of two linearly independent solutions.

Let f be nontrivial solution to equation

$$y'' + p(x)y' + q(x)y = 0.$$

Let's try to find solution of the form

$$y(x) = v(x)f(x),$$

where $v(x)$ is an unknown function. Differentiating, we have

$$y' = v'f + vf',$$

$$y'' = v''f + 2v'f' + vf''.$$

Substituting these expression into equation gives

$$v''f + 2v'f' + vf'' + p(v'f + vf') + qvf = 0$$

or

$$fv'' + (2f' + pf)v' = 0.$$

Let's $w(x) = v'(x)$, then we have

$$fw' + (2f' + pf)w = 0,$$

separating the variables and integrating gives

$$w = \pm \frac{e^{-\int p dx}}{f^2},$$

which holds on any interval where $f(x) \neq 0$.

$$v' = \pm \frac{e^{-\int p dx}}{f^2},$$

$$v = \pm \int \frac{e^{-\int p(x) dx}}{[f(x)]^2}.$$

Section 4.5 **Homogeneous Linear Equations with Constant Coefficients**

For the equation

$$ay'' + by' + cy = 0, \tag{3}$$

where a, b, c are constants.

$$(ar^2 + br + c)e^{rx} = 0.$$

The associated auxiliary equation is

$$ar^2 + br + c = 0.$$

When $\sqrt{\mathbf{b^2 - 4ac}} > \mathbf{0}$, the auxiliary equation has two different real roots

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \qquad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

,

So, $y_1(x) = e^{r_1 x}$ and $y_2(x) = e^{r_2 x}$ are two linearly independent solutions to (3) and

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

is the general solution to (3).

If $\sqrt{\mathbf{b^2 - 4ac}} = \mathbf{0}$, then the auxiliary equation has a repeated root $r \in \mathbf{R}$, $r = -\frac{b}{2a}$. In this case, $y_1(x) = e^{rx}$ and $y_2(x) = xe^{rx}$ are two linearly independent solutions to (3) and

$$y(x) = c_1 e^{rx} + c_2 x e^{rx} = (c_1 + c_2 x)e^{rx}$$

is the general solution to (3).

Section 4.6 **Auxiliary Equation with Complex Roots**

Complex conjugate roots.

If the auxiliary equation has complex conjugate roots $\alpha \pm i\beta$, then two linearly independent solutions to (3) are $\mathrm{e}^{\alpha x}\cos\beta x$ and $\mathrm{e}^{\alpha x}\sin\beta x$ and a general solution is

$$y(x) = c_1\mathrm{e}^{\alpha x}\cos\beta x + c_2\mathrm{e}^{\alpha x}\sin\beta x,$$

where c_1 and c_2 are arbitrary constants.

Section 4.7 **Superposition and nonhomogeneous equations**

Theorem (representation of solutions for nonhomogeneous equations). Let $y_p(x)$ be particular solution to the nonhomogeneous equation

$$y'' + p(x)y' + q(x)y = g(x) \tag{4}$$

on the interval (a, b) and let $y_1(x)$ and $y_2(x)$ be linearly independent solutions on (a, b) of the corresponding homogeneous equation

$$y'' + p(x)y' + q(x)y = 0.$$

Then a **general solution** of (4) on the interval (a, b) can be expressed in form

$$y(x) = y_p(x) + c_1y_1(x) + c_2y_2(x). \tag{5}$$

Procedure for solving solving nonhomogeneous equations.

To solve $y'' + p(x)y' + q(x)y = g(x)$:

(a) Determine the general solution $c_1y_1(x) + c_2y_2(x)$ of the corresponding homogeneous equation.

(b) Find the particular solution $y_p(x)$ of the given nonhomogeneous equation.

(c) Form the sum of the particular solution and a general solution to the homogeneous equation

$$y(x) = y_p(x) + c_1y_1(x) + c_2y_2(x),$$

to obtain the general solution to the given equation.

Section 4.8 **Method of Undetermined Coefficients**

We give a simple procedure for finding a particular solution to the equation

$$ay'' + by' + cy = g(x),$$

when the nonhomogeneous term $g(x)$ is of a special form

$$g(x) = \mathrm{e}^{\alpha x}(P_{m_1}(x)\cos\beta x + Q_{m_2}(x)\sin\beta x),$$

where

$$P_{m_1}(x) = p_0x^{m_1} + p_1x^{m_1-1} + p_2x^{m_1-2} + \ldots + p_{m_1-1}x + p_{m_1}$$

is a polynomial of degree m_1 and

$$Q_{m_2}(x) = q_0 x^{m_2} + q_1 x^{m_2-1} + q_2 x^{m_2-2} + \ldots + q_{m_2-1} x + q_{m_2}$$

is a polynomial of degree m_2, $\alpha, \beta \in \mathbf{R}$.

To apply the method of undetermined coefficients, we first have to solve the auxiliary equation for the corresponding homogeneous equation

$$ar^2 + br + c = 0$$

.

Particular solutions to $ay'' + by' + cy = g(x)$

Type	$g(x)$	$y_p(x)$
(I)	$p_0 x^{m_1} + p_1 x^{m_1-1} + \ldots + p_{m_1}$	$x^s(Ax^{m_1} + Bx^{m_1-1} + \ldots + Dx + F)$
(II)	$d\mathrm{e}^{\alpha x}$	$x^s A\mathrm{e}^{\alpha x}$
(III)	$\mathrm{e}^{\alpha x}(p_0 x^{m_1} + p_1 x^{m_1-1} + \ldots + p_{m_1})$	$x^s \mathrm{e}^{\alpha x}(Ax^{m_1} + Bx^{m_1-1} + \ldots + Dx + F)$
(IV)	$d\cos\beta x + f\sin\beta x$	$x^s(A\cos\beta x + B\sin\beta x)$
(V)	$(p_0 x^{m_1} + p_1 x^{m_1-1} + \ldots + p_{m_1})\cos\beta x+$ $+(q_0 x^{m_2} + q_1 x^{m_2-1} + \ldots + q_{m_2})\sin\beta x$	$x^s\{(A_0 x^m + A_1 x^{m-1} + \ldots + A_m)\cos\beta x+$ $+(B_0 x^m + B_1 x^{m-1} + \ldots + B_m)\sin\beta x\}$
(VI)	$\mathrm{e}^{\alpha x}(d\cos\beta x + f\sin\beta x)$	$x^s \mathrm{e}^{\alpha x}(A\cos\beta x + B\sin\beta x)$
(VII)	$\mathrm{e}^{\alpha x}[(p_0 x^{m_1} + p_1 x^{m_1-1} + \ldots + p_{m_1})\cos\beta x+$ $+(q_0 x^{m_2} + q_1 x^{m_2-1} + \ldots + q_{m_2})\sin\beta x]$	$x^s \mathrm{e}^{\alpha x}[(A_0 x^m + A_1 x^{m-1} + \ldots + A_m)\cos\beta x+$ $+(B_0 x^m + B_1 x^{m-1} + \ldots + B_m)\sin\beta x]$

In this table $s = 0$, when $\alpha + i\beta$ is not a root to the auxiliary equation, $s = 1$, when $\alpha + i\beta$ is one of two roots to the auxiliary equation, and $s = 0$, when $\beta = 0$ and α is a repeated root to the auxiliary equation; $m = \max\{m_1, m_2\}$.

Section 4.9 **Variation of Parameters**

Consider the nonhomogeneous linear second order differential equation

$$y'' + p(x)y' + q(x)y = g(x). \tag{6}$$

Let $\{y_1(x), y_2(x)\}$ be a fundamental solution set to the corresponding homogeneous equation

$$y'' + p(x)y' + q(x)y = 0.$$

The general solution to this homogeneous equation is $y_h(x) = c_1 y_1(x) + c_2 y_2(x)$, where c_1 and c_2 are constants. To find a particular solution to (6) we assume that $c_1 = c_1(x)$ and $c_2 = c_2(x)$ are functions of x and we seek a particular solution $y_p(x)$ in form

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x).$$

To summarize, we can find $c_1(x)$ and $c_2(x)$ solving the system

$$\begin{cases} c_1'(x)y_1(x) + c_2'(x)y_2(x) = 0 \\ c_1'(x)y_1'(x) + c_2'(x)y_2'(x) = g(x) \end{cases}$$

for $c_1'(x)$ and $c_2'(x)$. Cramer's rule gives

$$c_1'(x) = \frac{-g(x)y_2(x)}{W[y_1, y_2](x)}, \qquad c_2'(x) = \frac{g(x)y_1(x)}{W[y_1, y_2](x)}.$$

Then

$$c_1(x) = \int \frac{-g(x)y_2(x)}{W[y_1, y_2](x)} dx, \qquad c_2(x) = \int \frac{g(x)y_1(x)}{W[y_1, y_2](x)} dx.$$

Section 4.11 **A closer look at free mechanical vibrations**

A dumped mass-spring oscillator consists of a mass m attached to a spring fixed at one end. Model for the motion of the mass is expressed by the initial value problem

$$my'' + by' + ky = F_{\text{external}}, \qquad y(0) = y_0, \quad y'(0) = v_0,$$

where m is a mass, b is the dumping coefficient, k is the stiffness.

Undumped free case: $b = F_{\text{external}} = 0$

The equation reduces to

$$my'' + ky = 0$$

or

$$y'' + \omega^2 y = 0,$$

where $\omega = \sqrt{\frac{k}{m}}$. The solution of this equation is

$$y(t) = C_1 \cos \omega t + C_2 \sin \omega t$$

or

$$y(t) = A \sin \omega t + \phi,$$

where $A = \sqrt{C_1^2 + C_2^2}$, $\tan \phi = \frac{C_1}{C_2}$.

The motion is periodic with
period $2\pi/\omega$
natural frequency $\omega/2\pi$
angular frequency ω
amplitude A.

Underdumped or oscillatory motion ($b^2 < 4mk$)
The solution to the equation

$$my'' + by' + ky = 0$$

is

$$y(t) = \mathrm{e}^{\alpha t}(C_1 \cos \beta t + C_2 \sin \beta t) = A \sin \beta t + \phi,$$

where
$\alpha = -\frac{b}{2m}$, $\beta = \frac{1}{2m}\sqrt{4mk - b^2}$, $A = \sqrt{C_1^2 + C_2^2}$, $\tan \phi = \frac{C_1}{C_2}$.

Overdumped motion ($b^2 > 4mk$)

The solution to the equation

$$my'' + by' + ky = 0$$

is

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t},$$

where $r_1 = -\frac{b}{2m} + \frac{1}{2m}\sqrt{4mk - b^2}$, $r_1 = -\frac{b}{2m} - \frac{1}{2m}\sqrt{4mk - b^2}$

Critically dumped motion ($b^2 = 4mk$)
The solution to the equation

$$my'' + by' + ky = 0$$

is

$$y(t) = (c_1 + c_2 t)e^{-\frac{b}{2m}t}.$$

Section 4.12 **A closer look at forced mechanical vibrations**

Let's investigate the effect of a cosine forcing function on the system governed by the differential equation

$$my'' + by' + ky = F_0 \cos \gamma t,$$

where F_0, γ are nonnegative constants and $b^2 < 4mk$.
The general solution to this equation is

$$y(t) = Ae^{-(b/2m)t} \sin\left(\frac{\sqrt{4mk - b^2}}{2m} t + \phi\right) + \frac{F_0}{\sqrt{(k - m\gamma^2)^2 + b^2\gamma^2}} \sin(\gamma t + \theta),$$

where A, ϕ are constants and $\tan\theta = \frac{k - m\gamma^2}{b\gamma}$.

When the mass-spring system is **hung vertically**, the gravitational force can be ignored if $y(t)$ is measured from the equilibrium position. The equation of motion for this system is

$$my'' + by' + ky = F_{\text{external}},$$

where m is a mass, b is the dumping coefficient, k is the stiffness.

Chapter 5. Introduction to systems and phase plane analysis

Section 5.1 **Interconnected fluid tanks**

Two large tanks each holding m L of liquid, are interconnected by pipes, with the liquid flowing from tank A into tank B at a rate of a L/min and from B to A at a rate of b L/min. The liquid inside each tank is kept well stirred. A brine solution with a concentration of c kg/L of salt flows into tank A at a rate of d L/min. The solution flows out of the system, from tank A at e L/min and from tank B at f L/min. Initially, tank A contains g kg of salt and tank B contains h kg of salt. Determine the mass of salt at each tank at time $t > 0$.

SOLUTION. Let $x(t)$ be the mass of salt in tank A and $y(t)$ be the mass of salt in tank B. The salt *concentration* in tank A is $x(t)/m$ kg/L and in tank B is $y(t)/m$.

For tank A

$$\frac{dx}{dt} = \text{input rate} - \text{output rate}.$$

$$\text{input rate} = cd + \frac{dy(t)}{m}$$

$$\text{output rate} = \frac{ax(t)}{m} + \frac{ex(t)}{m}$$

Thus,

$$\frac{dx}{dt} = cd + d\frac{y}{m} - (a+e)\frac{x}{m}$$

For tank B

$$\frac{dy}{dt} = \text{input rate} - \text{output rate.}$$

$$\text{input rate} = a\frac{x(t)}{m}$$

$$\text{output rate} = \frac{by(t)}{m} + \frac{fy(t)}{m}$$

Thus,

$$\frac{dy}{dt} = a\frac{x}{m} - (b+f)\frac{y}{m}.$$

The IVP

$$\begin{cases} \frac{dx}{dt} = cd + \frac{d}{m}y - \frac{a+e}{m}x \\ \frac{dy}{dt} = \frac{a}{m}x - \frac{b+f}{m}y \end{cases}$$

$$x(0) = g, \quad y(0) = h$$

is a mathematical model for the mixing problem for the interconnected fluid tanks.

Section 5.2 **Elimination method for systems with constant coefficients**

Elimination procedure for 2×2 systems

To find a general solution to the system

$$\begin{cases} \frac{dx}{dt} = a_1x(t) + a_2y(t) + f_1(t) \\ \frac{dy}{dt} = b_1x(t) + b_2y(t) + f_2(t) \end{cases}$$

(a) Solve the second equation in system for $x(t)$ (or the first equation in system for $y(t)$).

$$x(t) = \frac{1}{b_1}(y' - b_2y(t) - f_2(t))$$

(b) Substitute the expression for $x(t)$ (or for $y(t)$) into another equation.

$$\frac{1}{b_1}(y'' - b_2y' - f_2'(t)) = \frac{a_1}{b_1}(y' - b_2y - f_2(t)) + f_1(t)$$

or

$$y'' - (a_1 + b_2)y' + (a_1b_2 - a_2b_1)y = f_2'(t) - a_1f_2(t) + b_1f_1(t) \tag{7}$$

(c) Find the general solution to (7).

(d) Substitute $y(t)$ into expression for $x(t)$.

Chapter 7. Laplace transform.

Section 7.2 **Definition of the Laplace Transform.**

Definition Let $f(x)$ be a function on $[0,\infty)$. The **Laplace transform** of f is the function F defined by the integral

$$F(s) = \int_0^\infty f(t)\mathrm{e}^{-st}dt.$$

The domain of $F(s)$ is all the values of s for which integral exists. The Laplace transform of f is denoted by both F and $\mathcal{L}\{f\}$.

Notice, that integral in definition is **improper** integral.

$$\int_0^\infty f(t)\mathrm{e}^{-st}dt = \lim_{N\to\infty}\int_0^N f(t)\mathrm{e}^{-st}dt$$

whenever the limit exists.

Table of Laplace transform

$f(t) = \mathcal{L}^{-1}\{F\}(t)$	$F(s) = \mathcal{L}\{f\}(s)$
1	$\frac{1}{s}, \quad s > 0$
e^{at}	$\frac{1}{s-a}, \quad s > a$
$t^n, \quad n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}}, \quad s > 0$
$\sin bt$	$\frac{b}{s^2+b^2}, \quad s > 0$
$\cos bt$	$\frac{s}{s^2+b^2}, \quad s > 0$
$\mathrm{e}^{at}t^n, \quad n = 1, 2, \ldots$	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
$\mathrm{e}^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
$\mathrm{e}^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
$u(t-a)$	$\frac{\mathrm{e}^{-as}}{s}$

Section 7.3 **Properties of Laplace transform.**

$$\mathcal{L}\{f+g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\}$$
$$\mathcal{L}\{cf\} = c\mathcal{L}\{f\} \text{ for any constant } c$$
$$\mathcal{L}\{\mathrm{e}^{at}f\}(s) = F(s-a)$$
$$\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$$
$$\mathcal{L}\{f''\}(s) = s^2\mathcal{L}\{f\}(s) - sf(0) - f'(0)$$
$$\mathcal{L}\{f^{(n)}\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0)$$
$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n\frac{d^n}{ds^n}(\mathcal{L}\{f(t)\})(s)$$
$$\mathcal{L}\{g(t)u(t-a)\}(s) = \mathrm{e}^{-as}\mathcal{L}\{g(t+a)\}(s)$$
$$\mathcal{L}^{-1}\{\mathrm{e}^{-as}\mathcal{L}\{f(t)\}(s)\} = f(t-a)u(t-a)$$

Section 7.4 **Inverse Laplace Transform.**

Definition Given a function $F(s)$, if there is a function $f(t)$ that is continuous on $[0, \infty)$ and satisfies

$$\mathcal{L}\{f\}(s) = F(s),$$

then we say that $f(t)$ is the **inverse Laplace transform** of $F(s)$ and employ the notation $f(t) = \mathcal{L}^{-1}\{F\}(t)$.

Section 7.5 **Solving initial value problems.**

To solve an initial value problem:

(a) Take the Laplace transform of both sides of the equation.

(b) Use the properties of the Laplace transform and the initial conditions to obtain an equation for the Laplace transform of the solution and then solve this equation for the transform.

(c) Determine the inverse Laplace transform of the solution.

Section 7.6 **Transforms of discontinuous and periodic functions**

Definition. The **unit step function** $u(t)$ is defined by

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

In general case

$$u(t-a) = \begin{cases} 0, & t < a \\ 1, & t > a. \end{cases}$$

The Laplace transform of $u(t-a)$ is

$$\mathcal{L}\{u(t-a)\} = \frac{\mathrm{e}^{-as}}{s}.$$

The properties of Laplace transform:

$$\mathcal{L}\{g(t)u(t-a)\}(s) = \mathrm{e}^{-as}\mathcal{L}\{g(t+a)\}(s)$$

$$\mathcal{L}^{-1}\{\mathrm{e}^{-as}\mathcal{L}\{f(t)\}(s)\} = f(t-a)u(t-a)$$

Chapter 9. Matrix methods for linear systems.

Section 9.1 **Introduction**

If a system of differential equations is expressed as

$$\begin{cases} x_1' = a_{11}(t)x_1 + a_{12}(t)x_2(t) + \ldots + a_{1n}(t)x_n + f_1(t) \\ x_2' = a_{21}(t)x_1 + a_{22}(t)x_2(t) + \ldots + a_{2n}(t)x_n + f_2(t) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_n' = a_{n1}(t)x_1 + a_{n2}(t)x_2(t) + \ldots + a_{nn}(t)x_n + f_n(t) \end{cases}$$

it is said to be a linear nonhomogeneous system in normal form. The matrix formulation of such a system is then

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(\mathbf{t}),$$

where $A(t)$ is the coefficient matrix

$$A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix}$$

$\mathbf{x}(\mathbf{t})$ is the solution vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{x}' = \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix}, \quad \mathbf{f}(\mathbf{t}) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}$$

An nth order linear differential equation

$$y^{(n)}(t) + p_{n-1}(t)y^{(n-1)}(t) + \cdots + p_0(t)y(t) = g(t)$$

can be rewritten as a first order system in normal form using the substitution $x_1(t) = y(t)$, $x_2(t) = y'(t)$,..., $x_n(t) = y^{(n-1)}(t)$

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_n' = -p_0(t)x_1 - p_1(t)x_2 - \ldots - p_{n-1}(t)x_n - g(t) \end{cases}$$

Section 9.4 **Linear system in normal form**

The system of n linear differential equations is in **normal form** if it is expressed as

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(\mathbf{t}),$$

where $A(t)$ is the coefficient $n \times n$ matrix, $\mathbf{x}(\mathbf{t}) = col(x_1(t), x_2(t), \ldots, x_n(t))$,
$\mathbf{f}(\mathbf{t}) = col(f_1(t), f_2(t), \ldots, f_n(t))$.

The **initial value problem** for the linear system is the problem for finding a differentiable vector function $\mathbf{x}(t)$ that satisfies the system on the interval I and also satisfies the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0 = col(x_{1,0}, ..., x_{n,0})$.

Theorem. Suppose $A(t)$ and $\mathbf{f}(t)$ are continuous on an open interval I that contains the point t_0. Then, for any choice of the initial vector $\mathbf{x}_0$, there exists a unique solution $\mathbf{x}(t)$ on the whole interval I to the IVP

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{f}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0.$$

Definition. The m vector functions $\mathbf{x}_1$,...,$\mathbf{x}_m$ are said to be **linearly dependent** (LD) on an interval I if there exist constants $c_1, ..., c_m$ not all zero, s.t.

$$c_1\mathbf{x}_1 + \ldots + c_m\mathbf{x}_m = \mathbf{0}$$

for all t in I. If the vectors are not linearly dependent, they are said to be **linearly independent** (LI) on I.

Definition. The **Wronskian** of n vectors functions $\mathbf{x}_1(t) = col(x_{1,1}, ..., x_{n,1})$,...,$\mathbf{x}_n(t) = col(x_{1,n}, ..., x_{n,n})$ is defined to be the real-valued function

$$W[\mathbf{x}_1, ..., \mathbf{x}_n](t) = \begin{vmatrix} x_{1,1}(t) & x_{1,2}(t) & \cdots & x_{1,n}(t) \\ x_{2,1}(t) & x_{2,2}(t) & \cdots & x_{2,n}(t) \\ \vdots & \vdots & \cdots & \vdots \\ x_{n,1}(t) & x_{n,2}(t) & \cdots & x_{n,n}(t) \end{vmatrix}$$

The Wronskian of solutions to the linear homogeneous system $\mathbf{x}'(t) = A(t)\mathbf{x}(t)$ is either identically zero or never zero on I. A set of n solutions $\mathbf{x}_1, ..., \mathbf{x}_n$ to $\mathbf{x}'(t) = A(t)\mathbf{x}(t)$ on I is LI on I if and only if their Wronskian is never zero on I.

Presentation of solutions (homogeneous case)
Let $\mathbf{x}_1, ..., \mathbf{x}_n$ be n LI particular solutions to the homogeneous system

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t)$$

on I. Then every solution to the system on I can be expressed in the form

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + ... + \mathbf{x}_n(t),$$

where $c_1, ..., c_n$ are constants.

A set of solutions $\{\mathbf{x}_1(t), ..., \mathbf{x}_n(t)\}$ that are LI is called a **fundamental solution set** (FSS) for $\mathbf{x}'(t) = A(t)\mathbf{x}(t)$. The linear combination $\mathbf{x}(t) = c_1\mathbf{x}_1(t) + ... + \mathbf{x}_n(t)$ is called a **general solution** to $\mathbf{x}'(t) = A(t)\mathbf{x}(t)$. A matrix

$$\mathbf{X}(t) = [\mathbf{x}_1(t), \mathbf{x}_2(t), ..., \mathbf{x}_n(t)] = \begin{bmatrix} x_{1,1}(t) & x_{1,2}(t) & \ldots & x_{1,n}(t) \\ x_{2,1}(t) & x_{2,2}(t) & \ldots & x_{2,n}(t) \\ \vdots & \vdots & \ldots & \vdots \\ x_{n,1}(t) & x_{n,2}(t) & \ldots & x_{n,n}(t) \end{bmatrix}$$

is called a **fundamental matrix** (FM) for $\mathbf{x}'(t) = A(t)\mathbf{x}(t)$.

Section 9.5 **Homogeneous linear system with constant coefficients.**

We want to obtain a general solution to the system

$$\mathbf{x}'(t) = A\mathbf{x}(t),$$

where A is a constant $n \times n$ matrix. We will be seeking particular solutions to the given system of the form

$$\mathbf{x}(t) = \mathrm{e}^{rt}\mathbf{u},$$

where r is a constant and $\mathbf{u}$ is a constant vector, both of which must be determined. Substituting $\mathbf{x}(t) = \mathrm{e}^{rt}\mathbf{u}$ in system gives

$$\mathrm{e}^{rt}\mathbf{u} = A\mathrm{e}^{rt}\mathbf{u} = \mathrm{e}^{rt}A\mathbf{u}$$

or

$$(A - r\mathbf{I})\mathbf{u} = \mathbf{0},$$

where

$$r\mathbf{I} = \begin{bmatrix} r & 0 & \dots & 0 \\ 0 & r & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & r \end{bmatrix}$$

Definition. Let $A = [a_{ij}]$ be an $n \times n$ constant matrix. The **eigenvalues** of A are those numbers r for which $(A - r\mathbf{I})\mathbf{u} = \mathbf{0}$ has at least one nontrivial solution $\mathbf{u}$. The corresponding nontrivial solutions $\mathbf{u}$ are called the **eigenvectors** of A associated with r.

System $(A - r\mathbf{I})\mathbf{u} = \mathbf{0}$ have a nontrivial solution if an only if

$$|A - r\mathbf{I}| = 0.$$

Equation $|A - r\mathbf{I}| = 0$ is called the **characteristic equation** of A.

Theorem. Suppose the $n \times n$ constant matrix A has n LI eigenvectors $\mathbf{u}_1$, $\mathbf{u}_2$,...,$\mathbf{u}_n$. Let r_i be the eigenvalue corresponding to $\mathbf{u}_i$. Then

$$\{\mathrm{e}^{r_1 t}\mathbf{u}_1, \mathrm{e}^{r_2 t}\mathbf{u}_2, ..., \mathrm{e}^{r_n t}\mathbf{u}_n\}$$

is a FSS on $(-\infty, +\infty)$ for the homogeneous system $\mathbf{x}'(t) = A\mathbf{x}(t)$. Consequently, a general solution to $\mathbf{x}'(t) = A\mathbf{x}(t)$ is

$$\mathbf{x}(t) = c_1\mathrm{e}^{r_1 t}\mathbf{u}_1 + c_2\mathrm{e}^{r_2 t}\mathbf{u}_2 + ... + c_n\mathrm{e}^{r_n t}\mathbf{u}_n,$$

where $c_1, c_2, ..., c_n$ are arbitrary constants.

Theorem. If r_1,...,r_m are distinct eigenvalues for the matrix A and $\mathbf{u}_i$ is an eigenvector associated with r_i, then $\mathbf{u}_1$,...,$\mathbf{u}_m$ are LI.

Corollary. If the $n \times n$ constant matrix A has n distinct eigenvalues and $\mathbf{u}_i$ is an eigenvector associated with r_i, then

$$\{\mathrm{e}^{r_1 t}\mathbf{u}_1, \mathrm{e}^{r_2 t}\mathbf{u}_2, ..., \mathrm{e}^{r_n t}\mathbf{u}_n\}$$

is a FSS for the homogeneous system $\mathbf{x}'(t) = A\mathbf{x}(t)$.

Linear Algebra

Matrices

Definition. An **m-by-n matrix** is a rectangular array of numbers that has m rows and n columns:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

Notation: $A = (a_{ij})_{1\le i\le n,\, 1\le j\le m}$, or simply $A = (a_{ij})$ if the dimensions are known.

An n-dimensional vector can be represented as a $1 \times n$ matrix (row vector) or as an $n \times 1$ matrix (column vector):

$$(x_1, x_2, \dots, x_n) \qquad \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

An $m \times n$ matrix $A = (a_{ij})$ can be regarded as a column of n-dimensional row vectors or as a row of m-dimensional column vectors:

$$A = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_m \end{pmatrix}, \qquad \mathbf{v}_i = (a_{i1}, a_{i2}, \dots, a_{in})$$

$$A = (\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n), \qquad \mathbf{w}_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$$

Matrix algebra

Definition. Let $A = (a_{ij})$ and $B = (b_{ij})$ be $m\times n$ matrices. The **sum** $A + B$ is defined to be the $m\times n$ matrix $C = (c_{ij})$ such that $c_{ij} = a_{ij} + b_{ij}$ for all indices i, j.

That is, two matrices with the same dimensions can be added by adding their corresponding entries.

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \\ a_{31}+b_{31} & a_{32}+b_{32} \end{pmatrix}$$

Definition. Given an $m\times n$ matrix $A = (a_{ij})$ and a number r, the **scalar multiple** rA is defined to be the $m\times n$ matrix $D = (d_{ij})$ such that $d_{ij} = ra_{ij}$ for all indices i, j.

That is, to multiply a matrix by a scalar r,
one multiplies each entry of the matrix by r.

$$r\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} ra_{11} & ra_{12} & ra_{13} \\ ra_{21} & ra_{22} & ra_{23} \\ ra_{31} & ra_{32} & ra_{33} \end{pmatrix}$$

The $m\times n$ **zero matrix** (all entries are zeros) is denoted O_{mn} or simply O.

Negative of a matrix: $-A$ is defined as $(-1)A$.

Matrix **difference**: $A - B$ is defined as $A + (-B)$.

Examples

$A = \begin{pmatrix} 3 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix}$, $\quad B = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$,

$C = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, $\quad D = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

$A + B = \begin{pmatrix} 5 & 2 & 0 \\ 1 & 2 & 2 \end{pmatrix}$, $\qquad A - B = \begin{pmatrix} 1 & 2 & -2 \\ 1 & 0 & 0 \end{pmatrix}$,

$2C = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$, $\qquad 3D = \begin{pmatrix} 3 & 3 \\ 0 & 3 \end{pmatrix}$,

$2C + 3D = \begin{pmatrix} 7 & 3 \\ 0 & 5 \end{pmatrix}$, $\qquad A + D$ is not defined.

Properties of linear operations

$(A + B) + C = A + (B + C)$

$A + B = B + A$

$A + O = O + A = A$

$A + (-A) = (-A) + A = O$

$r(sA) = (rs)A$

$r(A + B) = rA + rB$

$(r + s)A = rA + sA$

$1A = A$

$0A = O$

Matrix multiplication

The product of matrices A and B is defined if the number of columns in A matches the number of rows in B.

Definition. Let $A = (a_{ik})$ be an $m\times n$ matrix and $B = (b_{kj})$ be an $n\times p$ matrix.
The **product** AB is defined to be the $m\times p$ matrix $C = (c_{ij})$ such that $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$ for all indices i, j.

That is, matrices are multiplied **row by column**:

$$\begin{pmatrix} * & * & * \\ \boxed{*} & \boxed{*} & \boxed{*} \end{pmatrix} \begin{pmatrix} * & * & \boxed{*} & * \\ * & * & \boxed{*} & * \\ * & * & \boxed{*} & * \end{pmatrix} = \begin{pmatrix} * & * & * & * \\ * & * & \boxed{*} & * \end{pmatrix}$$

$$A = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_m \end{pmatrix}, \qquad \mathbf{v}_i = (a_{i1}, a_{i2}, \ldots, a_{in});$$

$$B = (\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_p), \qquad \mathbf{w}_j = \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{pmatrix}.$$

$$\Longrightarrow \ AB = \begin{pmatrix} \mathbf{v}_1{\cdot}\mathbf{w}_1 & \mathbf{v}_1{\cdot}\mathbf{w}_2 & \ldots & \mathbf{v}_1{\cdot}\mathbf{w}_p \\ \mathbf{v}_2{\cdot}\mathbf{w}_1 & \mathbf{v}_2{\cdot}\mathbf{w}_2 & \ldots & \mathbf{v}_2{\cdot}\mathbf{w}_p \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{v}_m{\cdot}\mathbf{w}_1 & \mathbf{v}_m{\cdot}\mathbf{w}_2 & \ldots & \mathbf{v}_m{\cdot}\mathbf{w}_p \end{pmatrix}.$$

Examples.

$$(x_1, x_2, \ldots, x_n) \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \left(\sum_{k=1}^{n} x_k y_k \right)$$

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} (x_1, x_2, \ldots, x_n) = \begin{pmatrix} y_1x_1 & y_1x_2 & \ldots & y_1x_n \\ y_2x_1 & y_2x_2 & \ldots & y_2x_n \\ \vdots & \vdots & \ddots & \vdots \\ y_nx_1 & y_nx_2 & \ldots & y_nx_n \end{pmatrix}$$

.

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 3 & 1 & 1 \\ -2 & 5 & 6 & 0 \\ 1 & 7 & 4 & 1 \end{pmatrix} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{-1} \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{0} & 3 & 1 & 1 \\ \mathbf{-2} & 5 & 6 & 0 \\ \mathbf{1} & 7 & 4 & 1 \end{pmatrix} = \begin{pmatrix} \square & * & * & * \\ * & * & * & * \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{-1} \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{0} & 3 & 1 & 1 \\ \mathbf{-2} & 5 & 6 & 0 \\ \mathbf{1} & 7 & 4 & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{-3} & * & * & * \\ * & * & * & * \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{-1} \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & \mathbf{3} & 1 & 1 \\ -2 & \mathbf{5} & 6 & 0 \\ 1 & \mathbf{7} & 4 & 1 \end{pmatrix} = \begin{pmatrix} -3 & \square & * & * \\ * & * & * & * \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{-1} \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & \mathbf{3} & 1 & 1 \\ -2 & \mathbf{5} & 6 & 0 \\ 1 & \mathbf{7} & 4 & 1 \end{pmatrix} = \begin{pmatrix} -3 & \mathbf{1} & * & * \\ * & * & * & * \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{-1} \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 3 & \mathbf{1} & 1 \\ -2 & 5 & \mathbf{6} & 0 \\ 1 & 7 & \mathbf{4} & 1 \end{pmatrix} = \begin{pmatrix} -3 & 1 & \boxed{} & * \\ * & * & * & * \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{-1} \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 3 & \mathbf{1} & 1 \\ -2 & 5 & \mathbf{6} & 0 \\ 1 & 7 & \mathbf{4} & 1 \end{pmatrix} = \begin{pmatrix} -3 & 1 & \mathbf{3} & * \\ * & * & * & * \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{-1} \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 3 & 1 & \mathbf{1} \\ -2 & 5 & 6 & \mathbf{0} \\ 1 & 7 & 4 & \mathbf{1} \end{pmatrix} = \begin{pmatrix} -3 & 1 & 3 & \boxed{} \\ * & * & * & * \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{-1} \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 3 & 1 & \mathbf{1} \\ -2 & 5 & 6 & \mathbf{0} \\ 1 & 7 & 4 & \mathbf{1} \end{pmatrix} = \begin{pmatrix} -3 & 1 & 3 & \mathbf{0} \\ * & * & * & * \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & -1 \\ \mathbf{0} & \mathbf{2} & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{0} & 3 & 1 & 1 \\ \mathbf{-2} & 5 & 6 & 0 \\ \mathbf{1} & 7 & 4 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 1 & 3 & 0 \\ \boxed{} & * & * & * \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & -1 \\ \mathbf{0} & \mathbf{2} & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{0} & 3 & 1 & 1 \\ \mathbf{-2} & 5 & 6 & 0 \\ \mathbf{1} & 7 & 4 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 1 & 3 & 0 \\ \mathbf{-3} & * & * & * \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & -1 \\ \mathbf{0} & \mathbf{2} & \mathbf{1} \end{pmatrix} \begin{pmatrix} 0 & \mathbf{3} & 1 & 1 \\ -2 & \mathbf{5} & 6 & 0 \\ 1 & \mathbf{7} & 4 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 1 & 3 & 0 \\ -3 & \boxed{} & * & * \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & -1 \\ \mathbf{0} & \mathbf{2} & \mathbf{1} \end{pmatrix} \begin{pmatrix} 0 & \mathbf{3} & 1 & 1 \\ -2 & \mathbf{5} & 6 & 0 \\ 1 & \mathbf{7} & 4 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 1 & 3 & 0 \\ -3 & \mathbf{17} & * & * \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & -1 \\ \mathbf{0} & \mathbf{2} & \mathbf{1} \end{pmatrix} \begin{pmatrix} 0 & 3 & \mathbf{1} & 1 \\ -2 & 5 & \mathbf{6} & 0 \\ 1 & 7 & \mathbf{4} & 1 \end{pmatrix} = \begin{pmatrix} -3 & 1 & 3 & 0 \\ -3 & 17 & \boxed{} & * \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & -1 \\ \mathbf{0} & \mathbf{2} & \mathbf{1} \end{pmatrix} \begin{pmatrix} 0 & 3 & \mathbf{1} & 1 \\ -2 & 5 & \mathbf{6} & 0 \\ 1 & 7 & \mathbf{4} & 1 \end{pmatrix} = \begin{pmatrix} -3 & 1 & 3 & 0 \\ -3 & 17 & \mathbf{16} & * \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & -1 \\ \mathbf{0} & \mathbf{2} & \mathbf{1} \end{pmatrix} \begin{pmatrix} 0 & 3 & 1 & \mathbf{1} \\ -2 & 5 & 6 & \mathbf{0} \\ 1 & 7 & 4 & \mathbf{1} \end{pmatrix} = \begin{pmatrix} -3 & 1 & 3 & 0 \\ -3 & 17 & 16 & \boxed{} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & -1 \\ \mathbf{0} & \mathbf{2} & \mathbf{1} \end{pmatrix} \begin{pmatrix} 0 & 3 & 1 & \mathbf{1} \\ -2 & 5 & 6 & \mathbf{0} \\ 1 & 7 & 4 & \mathbf{1} \end{pmatrix} = \begin{pmatrix} -3 & 1 & 3 & 0 \\ -3 & 17 & 16 & \mathbf{1} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 3 & 1 & 1 \\ -2 & 5 & 6 & 0 \\ 1 & 7 & 4 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 1 & 3 & 0 \\ -3 & 17 & 16 & 1 \end{pmatrix}$$

Any system of linear equations can be rewritten as a matrix equation.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

$$\iff \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

Properties of matrix multiplication:

$(AB)C = A(BC)$ (associative law)

$(A+B)C = AC + BC$ (distributive law #1)

$C(A+B) = CA + CB$ (distributive law #2)

$(rA)B = A(rB) = r(AB)$

If A and B are $n\times n$ matrices, then both AB and BA are well defined $n\times n$ matrices. However, in general, $AB \neq BA$.

Example. Let $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Then $AB = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$, $BA = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$.

Inverse matrix

Definition. Let A be an $n\times n$ matrix. The **inverse** of A is an $n\times n$ matrix, denoted A^{-1}, such that

$$\boxed{AA^{-1} = A^{-1}A = I.}$$

If A^{-1} exists then the matrix A is called **invertible**. Otherwise A is called **singular**.

A convenient way to compute the inverse matrix A^{-1} is to merge the matrices A and I into one 3×6 matrix $(A\,|\,I)$, and apply elementary row operations to this new matrix.

$$A = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}$$

$$, \qquad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} (A \,|\, I) = \left(\begin{array}{ccc|ccc} 3 & -2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{array}\right) (A \,|\, I) = \left(\begin{array}{ccc|ccc} 3 & -2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{array}\right)$$

As soon as the left half of the 3×6 matrix is converted to the identity matrix, we have got the inverse matrix A^{-1} in the right half.

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{5} & 0 & \frac{2}{5} \\ 0 & 1 & 0 & \frac{2}{5} & 0 & \frac{3}{5} \\ 0 & 0 & 1 & -\frac{3}{5} & 1 & -\frac{2}{5} \end{array}\right)$$

Thus $\begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{3}{5} & 0 & \frac{2}{5} \\ \frac{2}{5} & 0 & \frac{3}{5} \\ -\frac{3}{5} & 1 & -\frac{2}{5} \end{pmatrix}.$

That is,

$$\begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{5} & 0 & \frac{2}{5} \\ \frac{2}{5} & 0 & \frac{3}{5} \\ -\frac{3}{5} & 1 & -\frac{2}{5} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} \frac{3}{5} & 0 & \frac{2}{5} \\ \frac{2}{5} & 0 & \frac{3}{5} \\ -\frac{3}{5} & 1 & -\frac{2}{5} \end{pmatrix} \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Determinants

Determinant is a scalar assigned to each square matrix.

Notation. The determinant of a matrix $A = (a_{ij})_{1\le i,j\le n}$ is denoted $\det A$ or

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}.$$

Principal property: $\det A = 0$ if and only if the matrix A is singular.

Definition. $\det(a) = a$, $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - \\ - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

$$+:\ \begin{pmatrix} \boxed{*} & * & * \\ * & \boxed{*} & * \\ * & * & \boxed{*} \end{pmatrix}, \begin{pmatrix} * & \boxed{*} & * \\ * & * & \boxed{*} \\ \boxed{*} & * & * \end{pmatrix}, \begin{pmatrix} * & * & \boxed{*} \\ \boxed{*} & * & * \\ * & \boxed{*} & * \end{pmatrix}.$$

$$-:\ \begin{pmatrix} * & * & \boxed{*} \\ * & \boxed{*} & * \\ \boxed{*} & * & * \end{pmatrix}, \begin{pmatrix} * & \boxed{*} & * \\ \boxed{*} & * & * \\ * & * & \boxed{*} \end{pmatrix}, \begin{pmatrix} \boxed{*} & * & * \\ * & * & \boxed{*} \\ * & \boxed{*} & * \end{pmatrix}.$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - $$
$$- a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \to \left(\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}\right)$$

$$+ \left(\begin{array}{ccc|cc} \boxed{1} & \boxed{2} & \boxed{3} & * & * \\ * & \boxed{1} & \boxed{2} & \boxed{3} & * \\ * & * & \boxed{1} & \boxed{2} & \boxed{3} \end{array}\right) - \left(\begin{array}{ccc|cc} * & * & \boxed{1} & \boxed{2} & \boxed{3} \\ * & \boxed{1} & \boxed{2} & \boxed{3} & * \\ \boxed{1} & \boxed{2} & \boxed{3} & * & * \end{array}\right)$$

This rule works **only** for 3×3 matrices!

Examples (2×2 matrices)

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1, \qquad \begin{vmatrix} 3 & 0 \\ 0 & -4 \end{vmatrix} = -12$$

Examples (3×3 matrices)

$$\begin{vmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{vmatrix} = 3 \cdot 0 \cdot 0 + (-2) \cdot 1 \cdot (-2) + 0 \cdot 1 \cdot 3 -$$
$$- 0 \cdot 0 \cdot (-2) - (-2) \cdot 1 \cdot 0 - 3 \cdot 1 \cdot 3 = 4 - 9 = -5,$$

$$\begin{vmatrix} 1 & 4 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{vmatrix} = 1 \cdot 2 \cdot 3 + 4 \cdot 5 \cdot 0 + 6 \cdot 0 \cdot 0 -$$
$$- 6 \cdot 2 \cdot 0 - 4 \cdot 0 \cdot 3 - 1 \cdot 5 \cdot 0 = 1 \cdot 2 \cdot 3 = 6.$$

Eigenvalues and eigenvectors

Definition. Let A be an $n \times n$ matrix. A number r is called an **eigenvalue** of the matrix A if $\boxed{A\mathbf{v} = r\mathbf{v}}$ for a nonzero column vector $\mathbf{v}$.

The vector $\mathbf{v}$ is called an **eigenvector** of A belonging to (or associated with) the eigenvalue r.

The zero vector is never considered an eigenvector.

Definition. $\det(A - rI) = 0$ is called the **characteristic equation** of the matrix A.

Eigenvalues r of A are roots of the characteristic equation. Associated eigenvectors of A are nonzero solutions of the equation $(A - rI)\mathbf{x} = \mathbf{0}$.

Theorem. Let $A = (a_{ij})$ be an $n \times n$ matrix. Then $\det(A - rI)$ is a polynomial of r of degree n:

$$\det(A - rI) = (-1)^n r^n + c_1 r^{n-1} + \cdots + c_{n-1} r + c_n.$$

Definition. The polynomial $p(r) = \det(A - rI)$ is called the **characteristic polynomial** of the matrix A.

Corollary Any $n \times n$ matrix has at most n eigenvalues.

Example. Find the eigenvalues and corresponding eigenvectors for the matrix

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

The characteristic polynomial for the matrix A is

$$|A - rI| = \begin{vmatrix} 2-r & -1 & 1 \\ 1 & 2-r & -1 \\ 1 & -1 & 2-r \end{vmatrix} =$$

$$= (2-r)^3 + 1 - 1 - (2-r) + (2-r) - (2-r) = (2-r)((2-r)^2 - 1) = (1-r)(2-r)(3-r)$$

Thus, the eigenvalues for the matrix A are

$$r_1 = 1, \quad r_2 = 2, \quad r_3 = 3.$$

The eigenvector $\mathbf{u}_1 = col(u_{11}, u_{12}, u_{13})$, associated with eigenvalue $r_1 = 1$, is a solution to the system

$$\begin{pmatrix} 2-1 & -1 & 1 \\ 1 & 2-1 & -1 \\ 1 & -1 & 2-1 \end{pmatrix} \begin{pmatrix} u_{11} \\ u_{12} \\ u_{13} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

or

$$\begin{cases} u_{11} - u_{12} + u_{13} = 0 \\ u_{11} + u_{12} - u_{13} = 0 \end{cases}$$

Adding two equation of this system gives

$$u_{11} = 0, \qquad u_{12} = u_{13} = c,$$

where c is an arbitrary constant. Since c is an arbitrary constant, we can choose $c = 1$. So, the eigenvector corresponding to eigenvalue $r_1 = 1$ is

$$\mathbf{u}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Similarly, the eigenvector $\mathbf{u}_2 = col(u_{21}, u_{22}, u_{23})$ corresponding to $r_2 = 2$ is a solution to the system

$$\begin{pmatrix} 2-2 & -1 & 1 \\ 1 & 2-2 & -1 \\ 1 & -1 & 2-2 \end{pmatrix} \begin{pmatrix} u_{21} \\ u_{22} \\ u_{23} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

or

$$\begin{cases} -u_{22} + u_{23} = 0 \\ u_{21} - u_{23} = 0 \\ u_{21} - u_{22} = 0 \end{cases}$$

The solution to this system is $u_{21} = u_{22} = u_{23} = 1$. Thus, the eigenvector corresponding to eigenvalue $r_2 = 2$ is

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

The eigenvector $\mathbf{u}_3 = col(u_{31}, u_{32}, u_{33})$ corresponding to $r_3 = 3$ is a solution to the system

$$\begin{pmatrix} 2-3 & -1 & 1 \\ 1 & 2-3 & -1 \\ 1 & -1 & 2-3 \end{pmatrix} \begin{pmatrix} u_{31} \\ u_{32} \\ u_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

or

$$\begin{cases} -u_{31} - u_{32} + u_{33} = 0 \\ u_{31} - u_{32} - u_{33} = 0 \end{cases}$$

Adding two equation of this system gives $u_{32} = 0, \quad u_{31} = u_{33} = 1$ so, the eigenvector corresponding to eigenvalue $r_3 = 3$ is

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$