7. If $W[f,g] = t^2e^t$ and $f(t) = t$, find $g(t)$.
SOLUTION.

$$W[f,g] = \begin{vmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{vmatrix} = \begin{vmatrix} t & g(t) \\ 1 & g'(t) \end{vmatrix} = tg'(t) - g(t) = t^2e^t$$

The differential equation for $g(t)$ is

$$tg'(t) - g(t) = t^2e^t$$

It is a linear first-order differential equation. Its standart form is

$$g'(t) - \frac{1}{t}g(t) = te^t$$

The differential equation for the integrating factor μ is

$$\mu'(t) + \frac{1}{t}\mu(t) = 0$$

$$\mu(t) = \frac{1}{t}$$

Then,

$$\frac{d}{dt}\left(\frac{1}{t}g(t)\right) = \frac{1}{t}te^t = e^t$$

$$\frac{1}{t}g(t) = \int e^t dt = e^t + C$$

Thus,

$$g(t) = te^t + Ct.$$