MATH 308-519 Spring 2011 Practice Final

1. A large tank initially contains 10 L of fresh water. A brine containing 20 g/L of salt flows into the tank at a rate of 3 L/min. The solution inside the tank is kept well stirred and flows out of the tank at the rate 2 L/min. Determine the concentration of salt in the tank as a function of time.

2. Suppose that a sum S_0 is invested at an annual rate of return r compounded continuously.

   (a) Find the time T required for the original sum to double in value as a function of r.

   (b) Determine T if $r = 7\%$.

   (c) Find the return rate that must be achieved if the initial investment is to double in 8 years.

3. An object with temperature 150^0 is placed in a freezer whose temperature is 30^0. Assume that the temperature of the freezer remains essentially constant.

   (a) If the object is cooled to 120^0 after 8 min, what will its temperature be after 18 min?

   (b) When will its temperature be 60^0?

4. Determine (without solving the problem) an interval in which the solution to the initial value problem
$$(4-t^2)y' + 2ty = 3t^2, \quad y(1) = -3$$
is certain to exist.

5. Solve the initial value problem
$$y' = \frac{t^2}{1+t^3}, \quad y(0) = y_0$$
and determine how the interval in which the solution exists depends on the initial value y_0.

6. Solve the following initial value problem
$$\sqrt{y}dt + (1+t)dy = 0 \quad y(0) = 1.$$

7. Find the general solution to the equation
$$(t^2-1)y' + 2ty + 3 = 0$$

8. Solve the initial value problem
$$(ye^{xy}\cos(2x) - 2e^{xy}\sin(2x) + 2x)dx + (xe^{xy}\cos(2x) - 3)dy = 0, \quad y(0) = -1$$

9. Solve the following systems:

   (a) $\begin{cases} 2x_1 - 3x_2 = 4 \\ x_1 + 2x_2 = -5 \end{cases}$

(b) $\begin{cases} x_1 - 3x_2 = 5 \\ -2x_1 + 6x_2 = 1 \end{cases}$

(c) $\begin{cases} -2x_1 + x_2 = 4 \\ 6x_1 - 3x_2 = -12 \end{cases}$

10. Find the general solution of the system of equations/solve the initial value problem:

    (a) $\vec{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \vec{x}, \qquad \vec{x}(0) = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

    (b) $\vec{x}' = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \vec{x}$

    (c) $\vec{x}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \vec{x}, \qquad \vec{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

11. Use variation of parameters to find the general solution for the system

$$\vec{x}' = \begin{pmatrix} -4 & -2 \\ 6 & 3 \end{pmatrix} \vec{x} + \begin{pmatrix} \dfrac{2}{e^t - 1} \\ -\dfrac{3}{e^t - 1} \end{pmatrix}$$

12. A mass of 20 g stretches a spring 5 cm. Suppose that the mass is also attached to a viscous damper with a damping constant of 400 dyne-s/cm. The mass is pulled down an additional 2 cm and then released. Formulate the initial value problem that governs the motion of the mass.

13. Determine which of the following operators are linear:

    (a) $L_1[y] = y'' + e^y y$
    (b) $L_2[y] = ty'' + t^2 y' + (t^3 - 1)y$
    (c) $L_3[y] = t(y'')^2 + y$
    (d) $L_4[y] = 2y'' + 3y' + 4y$
    (e) $L_5[y] = 2y'' + 2(y')^2 + 4y$

14. Determine the longest interval in which the solution to the initial value problem

$$(x-3)y'' + xy' + y \ln x = 0 \qquad y(1) = 0 \quad y'(1) = 1$$

    is certain to have a unique twice differentiable solution.

15. If $W[f, g] = t^2 e^t$ and $f(t) = t$, find $g(t)$.

16. Verify that the functions $y_1(x) = x$, $y_2(x) = \sin x$ are solutions of the differential equation

$$(1 - x \cot x)y'' - xy' + y = 0, \qquad 0 \le x \le \pi.$$

    Do they constitute a fundamental set of solutions?

17. Find the general solution of the equation:

(a) $y'' - 3y' + 2y = x\cos x$

(b) $y'' - 9y' = e^{-3x}(x^2 + \sin 3x)$

(c) $y'' - 2y' + y = \dfrac{e^x}{x}$

18. Solve the Cauchy-Euler equation

$$x^2y'' - 4xy' + 6y = 0$$

19. Given that $y_1(x) = x^{-2}$ is a solution to

$$x^2y'' + 6xy' + 6y = 0,$$

Find the second linearly independent solution to this equation.

20. Find $\mathcal{L}\{f(t)\}(s)$ if

$$f(t) = \begin{cases} t, & 0 \le t \le 1 \\ 3 - t, & 1 < t \le 2 \\ 1, & t > 2 \end{cases}$$

21. Find $\mathcal{L}\{t\cos t + e^{3t}\sin 2t + t^5e^{2t}\}$

22. Find $\mathcal{L}^{-1}\left\{\dfrac{3s+2}{(s^2-4)(s+1)}\right\}$

23. Solve the initial value problem using the method of Laplace transform

$$y'' + 2y' + y = 4e^{-t}, \quad y(0) = 2, \ \ y'(0) = -1$$

24. Find $\mathcal{L}\{f(t)\}$ if

$$f(t) = \begin{cases} 0, & 0 < t \le 1 \\ t, & 1 < t \le 2 \end{cases} \quad \text{and } f(t) \text{ has period } 2$$

25. Find $\mathcal{L}^{-1}\left\{\dfrac{e^{-3s}(s-5)}{(s+1)(s+2)}\right\}$.

26. For the system

$$\begin{cases} \dfrac{dx}{dt} = x - y \\ \dfrac{dy}{dt} = x - 2y + xy - 2 \end{cases}$$

(a) Determine all critical points.

(b) Find the corresponding linear system near each critical point.

(c) Determine whether each critical point is asymptotically stable, stable or unstable, and classify it as to type.