- Equation that contains some derivatives of an unknown function is called a **differential equation**.
- If an equation involves the derivative of one variable with respect to another, then the former is called a **dependent variable** and the latter is called an **independent variable**.
- A differential equation involving only ordinary derivatives with respect to a single variable is called an **ordinary differential equations** or ODE. A differential equation involving partial derivatives with respect to more then one variable is a **partial differential equations** or PDE.
- The **order** of a differential equation is the order of the highest-order derivatives present in equation.
- An ODE is **linear** if it has format
$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + a_1(x)\frac{dy}{dx} + a_0(x)y = F(x),$$
where $a_n(x)$, $a_{n-1}(x)$,...,$a_0(x)$ and $F(x)$ depend only on variable x. If an ODE is not linear, then we call it **nonlinear**.

Example 1. For each of the differential equations indicate whether it is linear or nonlinear.

1. $\ln(x)\dfrac{d^2y}{dx^2} + 3e^x\dfrac{dy}{dx} - y\sin x = 0$
2. $2y'' - 3y^2 = \mathrm{e}^x$
3. $\dfrac{d^3y}{dx^3} + (x^2-1)y + \cos x = 0$
4. $y'' - \sin(x+y)y' + (x^2+1)y = 0$
5. $y'' - e^{xy} = \cos(2x+y)$

A general form for an nth-order equation with variable x and unknown function $y = y(x)$ can be expressed as

$$F\left(x, y, \frac{dy}{dx}, \cdots, \frac{d^ny}{dx^n}\right) = 0, \tag{1}$$

where F is a function that depends on x, y, and the derivatives of y up to the order n. We assume that the equation holds for all x in an open interval I ($a < x < b$, where a or b could be infinite).

$$\frac{d^ny}{dx^n} = f\left(x, y, \frac{dy}{dx}, \cdots, \frac{d^{n-1}y}{dx^{n-1}}\right) = 0. \tag{2}$$

- A function $\varphi(x)$ that when substituted by y in the equation satisfies the equation for all x in the interval I is called a **solution** to the equation on I.

- By an **initial value problem** for an nth-order differential equation

$$F\left(x, y, \frac{dy}{dx}, \cdots, \frac{d^n y}{dx^n}\right) = 0$$

we mean: Find a solution to the differential equation on an interval I that satisfies at x_0 the n initial conditions:

$$y(x_0) = y_0, \quad \frac{dy}{dx}(x_0) = y_1, ..., \frac{d^{n-1}y}{dx^{n-1}}(x_0) = y_{n-1},$$

where $x_0 \in I$ and y_0, y_1,...,y_{n-1} are given constants.

- In case of a first-order equation $F\left(x, y, \frac{dy}{dx}\right)$, the initial conditions reduce to the single requirement

$$y(x_0) = y_0$$

- In case of a second-order equation, the initial conditions have the form

$$y(x_0) = y_0, \qquad \frac{dy}{dx}(x_0) = y_1.$$

Direction Fields.

One technique that is useful in graphing the solutions to a first-order differential equation is to sketch the direction field for the equation. To describe this method, we need to make a general observation.

Namely, a first-order equation

$$\frac{dy}{dx} = f(x, y)$$

specifies a slope at each point in the xy-plane where f is defined.

A plot of short line segments drawn at various points in the xy-plane showing the slope of the solution curve there is called a **direction field** for the differential equation. Because the direction field gives the "flow of solutions", it facilitates the drawing of any particular solution (such as the solution to an initial value problem).

Example 2.

1. Sketch a direction field for $y' = t + y$

2. Using the direction field, describe the behavior of the solution as $t \to \pm\infty$.

We can get directional field using MatLab.

```
>>[t,y]=meshgrid(-3:0.2:2, -1:0.2:5);
>> s=t+y;
>> quiver(t,y,ones(size(s)), s), axis tight
```

All array must have the same size, therefore we do

```
>>[t,y]=meshgrid(-3:0.2:2, -1:0.2:5);
>> s=t+y;
>> l=sqrt(1+s.∧ 2);
>> quiver(t,y,1./l,s./l,0.5), axis tight
```

To graph solutions, we do

```
>>[t,y]=meshgrid(-3:0.2:2, -1:0.2:5);
>> s=t+y;
>> l=sqrt(1+s.∧ 2);
>> quiver(t,y,1./l,s./l,0.5), axis tight
>> f=@(t,y)t+y;
>> hold on
>> for y0=-3:1:2
[ts,ys]=ode45(f,[-3,2],y0); plot(ts,ys)
end
>> hold off
```

Example 3.

1. Sketch a direction field for $y' = t^2 - y$
2. Using the direction field, describe the behavior of the solution as $t \to \pm\infty$.