1. The Existence and Uniqueness Theorem guarantees that the solution to

$$x^3 y'' + \frac{x}{\sin x} y' - \frac{2}{x-5} y = 0, \quad y(2) = 6, \quad y'(2) = 7$$

uniquely exists on

(a) $(-\pi, \pi)$
(b) $(0, \pi)$
(c) $(5, \infty)$
(d) $(0, 5)$

$$y'' + p(x) y' + q(x) y = g(x)$$
$$y(x_0) = y_0, \quad y'(x_0) = y_1$$

The initial value problem has a unique solution on the interval I such that:

- p, q, and g are continuous on I,
- x_0 is in I

$$\frac{x^3 y'' + \frac{x}{\sin x} y' - \frac{2}{x-5} y = 0}{x^3}$$

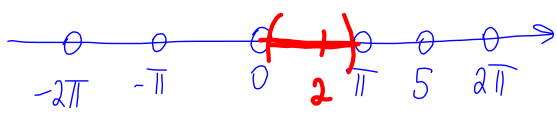
$$y'' + \frac{1}{x^2 \sin x} y' - \frac{2}{x^2(x-5)} y = 0$$

$$p(x) = \frac{1}{x^2 \sin x}$$

p is continuous if $x \neq 0$
$(\sin x \neq 0)$ $x \neq \pi n, \ n = 0, \pm 1, \pm 2, \ldots$

$$q(x) = -\frac{2}{x^2(x-5)}$$

q is continuous if $x \neq 0$
$x \neq 5$



2. All of the following pairs of functions form a fundamental set of solutions to some second order differential equation on $(-\infty, \infty)$ EXCEPT

(a) $1, \quad e^{-t}$

(b) $\cos t, \quad \sin(t + 2\pi)$

(c) $e^{-2t}\cos 2t, \quad e^{-2t}\sin 2t$

(d) $e^{5t}, \quad e^{5t-1}$

Find a pair of functions $\{y_1, y_2\}$ such that $W[y_1, y_2] = 0$.

1) $W[e^{at}, e^{bt}] \neq 0$ if $a \neq b$

2) $W[\cos bt, \sin bt] \neq 0$

3) $W[e^{at}\cos bt, e^{at}\sin bt] \neq 0$

$$W[e^{5t}, e^{5t-1}] = \begin{vmatrix} e^{5t} & e^{5t-1} \\ 5e^{5t} & 5e^{5t-1} \end{vmatrix}$$

$$= 5e^{5t}e^{5t-1} - 5e^{5t}e^{5t-1} = 0$$

3. Which of the following will be a particular solution to the equation

$$4y'' + 4y' + y = 24x\mathrm{e}^{\frac{x}{2}}?$$

(a) $x^2(Ax + B)\mathrm{e}^{\frac{x}{2}}$

(b) $(Ax + B)\mathrm{e}^{\frac{x}{2}}$

(c) $x(Ax + B)\mathrm{e}^{\frac{x}{2}}$

(d) $(Ax + B)\sin\frac{x}{2} + (Cx + D)\cos\frac{x}{2}$

4. A 2-kg mass is attached to a spring with stiffness $k = 50$ N/m. The damping force is negligible. What is the resonance frequency for the system?

(a) $5/2\pi$

(b) $2/2\pi$

(c) $3/2\pi$

(d) $4/2\pi$

$$2y'' + 50y = 0$$

$$y'' + 25y = 0$$

$$f_r = \frac{\sqrt{25}}{2\pi}$$

$$= \frac{5}{2\pi}$$

5. The motion of the mass-spring system with damping is governed by $y'' + 2y' + y = 0$, $y(0) = 1, y'(0) = -3$. This motion is

(a) undamped $(b=0)$

(b) underdamped $(b^2-4mk<0)$

(c) critically damped $(b^2-4mk=0)$

(d) overdamped $(b^2-4mk>0)$

$y''+2y'+y=0$

$m=1$

$b=2$

$k=1$

$b^2-4mk = 4-4=0$

6. $e^{2+\frac{3\pi}{4}i} =$

(a) π

(b) $\dfrac{\sqrt{2}}{2}(1+i)e^2$

(c) $\dfrac{\sqrt{2}}{2}(1-i)e^2$

(d) $\dfrac{\sqrt{2}}{2}(-1+i)e^2$

Euler's formula

$$e^{a+bi} = e^a(\cos b + i \sin b)$$

$$e^{2+\frac{3\pi}{4}i} = e^2\left(\cos\frac{3\pi}{4} + i \sin\frac{3\pi}{4}\right)$$

$$= e^2\left(-\frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}\right)$$

7. A 2-kg mass is attached to a spring with stiffness $k = 50$ N/m. The mass is displaced 1/4 m to the left of the equilibrium point and given a velocity of 1 m/sec to the left. The damping force is negligible. The amplitude of this vibration is

(a) $\frac{\sqrt{41}}{20}$

(b) 1

(c) $\frac{\sqrt{20}}{41}$

(d) $\frac{1}{4}$

$$\left.\begin{aligned} 2y'' + 50y &= 0 \\ y(0) &= -1/4 \\ y'(0) &= -1 \end{aligned}\right\} \text{IVP}$$

$$y'' + 25y = 0$$

$$y(t) = C_1 \cos 5t + C_2 \sin 5t$$

$$y(0) = \boxed{C_1 = -1/4}$$

$$y'(t) = -5C_1 \sin 5t + 5C_2 \cos 5t$$

$$y'(0) = 5C_2 = -1$$

$$\boxed{C_2 = -1/5}$$

$$\text{amplitude} = \sqrt{C_1^2 + C_2^2}$$

$$= \sqrt{\frac{1}{16} + \frac{1}{25}}$$

$$= \frac{\sqrt{16+25}}{4 \cdot 5}$$

8. The FSS to the equation $y'' - 2y' + 5y = 0$ is

(a) $\{\cos x, \sin x\}$
(b) $\{e^x \cos 2x, e^x \sin 2x\}$
(c) $\{e^x, xe^x\}$
(d) $\{e^x, e^{-x}\}$

auxiliary eqn:

$r^2 - 2r + 5 = 0$

$$r_1 = \frac{2 + \sqrt{4-20}}{2}$$

$$= \frac{2 + \sqrt{-16}}{2}$$

$$= 1 + 2i$$

general solution:

$$y(t) = e^t(c_1 \cos 2t + c_2 \sin 2t)$$

$$= c_1 e^t \cos 2t + c_2 e^t \sin 2t$$

$$FSS = \{e^t \cos 2t, e^t \sin 2t\}$$

9. Given that $y_1(x) = -\frac{1}{2}x^2 + \frac{1}{2}x - \frac{3}{4}$ is a solution to $y'' - y' - 2y = x^2$ and $y_2(x) = \frac{1}{4}e^{3x}$ is a solution to $y'' - y' - 2y = e^{3x}$. A solution to $y'' - y' - 2y = 2x^2 - e^{3x}$ is

(a) $-x^2 + x - \frac{3}{2} - \frac{1}{4}e^{3x}$

(b) $x^2 - x - \frac{3}{2} - \frac{1}{4}e^{3x}$

(c) $-x^2 + x + \frac{3}{4} - e^{3x}$

(d) $x - \frac{3}{2} - \frac{1}{4}e^{3x}$

$g_1(x) = x^2, \quad g_2(x) = e^{3x}$

$g_3(x) = 2x^2 - e^{3x}$
$= 2g_1 - g_2$

$y(x) = 2y_1 - y_2$

If $y_1(x)$ is a particular solution of

$$y'' + p(x)y' + q(x)y = g_1(x)$$

and $y_2(x)$ is a particular solution of

$$y'' + p(x)y' + q(x)y = g_2(x),$$

then $y(x) = Ay_1(x) + By_2(x)$ is a particular solution of

$$y'' + p(x)y' + q(x)y = Ag_1(x) + Bg_2(x)$$

10. The Wronskian of two functions $y_1(x) = x + 2x^2$ an $y_2(x) = 2^x$ is

(a) $2^x(1 + 4x - x(1 + 2x))$

(b) $-2^x(1 + 4x - x\ln 2(1 + 2x))$

(c) $(1 + 4x - x(1 + 2x))$

(d) $2^x(1 + 2x - x\ln 2(1 + 4x))$

$$W[y_1, y_2] = \begin{vmatrix} x+2x^2 & 2^x \\ 1+4x & 2^x \ln 2 \end{vmatrix}$$

$$= 2^x \ln 2(x+2x^2) - 2^x(1+4x)$$

$$= 2^x(\ln 2(x+2x^2) - 1-4x)$$

11. Find a general solution to the equation

$$y'' + 6y' + 9y = \frac{e^{-3x}}{1+2x}$$

variation of parameters.

homogeneous eqn:

$$y''+6y'+9y=0$$

$$r^2+6r+9=0$$

$$(r+3)^2=0$$

$$r=-3 \text{ - repeated root}$$

$$y_h(x)=(C_1+C_2x)e^{-3x}$$

$$= C_1\underbrace{e^{-3x}}_{y_1(x)} + C_2\underbrace{xe^{-3x}}_{y_2(x)}$$

$$y(x)= C_1(x)e^{-3x}+C_2(x)xe^{-3x}$$

$$\begin{cases} C_1'y_1+C_2'y_2=0 \\ C_1'y_1'+C_2'y_2'=g(x) \end{cases}$$

$$\begin{cases} C_1'\cancel{e^{-3x}}+C_2'x\cancel{e^{-3x}}=0 \\ C_1'(-3)\cancel{e^{-3x}}+C_2'\cancel{e^{-3x}}+C_2'x(-3)\cancel{e^{-3x}}=\frac{\cancel{e^{-3x}}}{1+2x} \end{cases}$$

$$\begin{cases} C_1'+C_2'x=0 \Rightarrow \boxed{C_1'=-C_2'x} \\ -3C_1'-3xC_2'+C_2'=\frac{1}{1+2x} \end{cases}$$

$$C_2'=\frac{1}{1+2x}$$

$$\boxed{C_2(x)=\frac{1}{2}\ln|1+2x|+C_3}$$

$$C_1'(x)=-xC_2'$$

$$=-\frac{x}{1+2x} \quad \text{- improper fraction}$$

$$=-\left(\frac{1}{2}-\frac{1}{2}\frac{1}{1+2x}\right)$$

$$C_1'(x)=-\frac{1}{2}+\frac{1}{2+4x}$$

$$\boxed{C_1(x)=-\frac{1}{2}x+\frac{1}{4}\ln|2+4x|+C_4}$$

$$\begin{array}{r} 1/2 \\ x\,\overline{)\,1+2x} \\ \underline{-x+1/2} \\ -1/2 \end{array}$$

$$\boxed{y(x)=\left(-\frac{1}{2}x+\frac{1}{4}\ln|2+4x|+C_4\right)e^{-3x}+\left(\frac{1}{2}\ln|1+2x|+C_3\right)xe^{-3x}}$$

general solution.

12. Find a general solution to the equation

$$4y'' + y' = 4x^3 + 48x^2 + 1$$

undetermined coefficients.

homogeneous eqn.

$4y'' + y' = 0$

$4r^2 + r = 0$

$r(4r+1) = 0$

$r_1 = 0, \quad r_2 = -1/4$

$y_h(x) = C_1 + C_2 e^{-\frac{1}{4}x}$

r=0 is a root of the auxiliary eqn

$y_p(x) = x(Ax^3 + Bx^2 + Cx + D)$

$= Ax^4 + Bx^3 + Cx^2 + Dx$

$y_p' = 4Ax^3 + 3Bx^2 + 2Cx + D$

$y_p'' = 12Ax^2 + 6Bx + 2C$

$4(12Ax^2 + 6Bx + 2C) + 4Ax^3 + 3Bx^2 + 2Cx + D = 4x^3 + 48x^2 + 1$

$x^3: \ 4A = 4 \qquad \boxed{A=1}$

$x^2: \ 48A + 3B = 48 \qquad 3B = 48 - 48A = 0 \qquad \boxed{B=0}$

$x: \ 24B + 2C = 0 \qquad \boxed{C=0}$

$1: \ 8C + D = 1 \qquad \boxed{D=1}$

$y_p(x) = x^4 + x$

$$\boxed{y(t) = C_1 + C_2 e^{-\frac{1}{4}x} + x^4 + x}$$

13. Given that $y_1(x) = x$ is a solution to

$$x^2y'' + xy' - y = 0,$$

find a second solution of this equation on $(0, +\infty)$.

<u>reduction of order</u>

unknown function

General solution: $y(x) = y_1(x)\,v(x)$

$= x\,v(x)$

$$y'(x) = v(x) + x v'(x)$$

$$y''(x) = v'(x) + v'(x) + x v''(x)$$

$$= 2v'(x) + x v''(x)$$

Plug y, y', y'' into the eqn:

$$x^2\left(2v'(x) + x v''(x)\right) + x\left(v(x) + x v'(x)\right) - x v(x) = 0$$

$$2x^2 v'(x) + x^3 v''(x) + \cancel{x v(x)} + x^2 v'(x) - \cancel{x v(x)} = 0$$

$$\frac{x^3 v''(x) + 3x^2 v'(x) = 0}{x^2}$$

$$x v''(x) + 3v'(x) = 0$$

substitution: $v'(x) = u(x)$

$v''(x) = u'(x)$

$$x u' + 3u = 0 \qquad x\frac{du}{dx} + 3u = 0$$

$$\int \frac{du}{u} = -3\int \frac{dx}{x}$$

$$\ln|u| = -3\ln|x| + \ln C$$

$$u(x) = C_1 x^{-3}$$

$$v'(x) = C_1 x^{-3}$$

$$v(x) = C_1 \frac{x^{-2}}{-2} + C_2$$

$$= C_3 x^{-2} + C_2 \qquad \boxed{C_3 = \frac{C_1}{-2}}$$

$$y(x) = (C_3 x^{-2} + C_2)x$$

$$= C_3 \underbrace{x^{-1}}_{y_2(x)} + C_2 \underbrace{x}_{y_1(x)}$$

$$\boxed{y_2(x) = x^{-1}}$$