

1. Find the Laplace transform of the given function.

$$(a) f(t) = \begin{cases} \frac{t}{2}, & 0 \leq t < 6 \\ 3, & t \geq 6 \end{cases}, \quad f(t) = \frac{t}{2} + \left(3 - \frac{t}{2}\right) u_6(t)$$

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \mathcal{L}\left\{\frac{t}{2} + \left(3 - \frac{t}{2}\right) u_6(t)\right\} \\ &= \mathcal{L}\left\{\frac{t}{2}\right\} + \mathcal{L}\left\{\frac{1}{2}(6-t) u_6(t)\right\} \\ &= \frac{1}{2s^2} + \frac{1}{2}(-1) \mathcal{L}\{(t-6) u_6(t)\} \\ &= \frac{1}{2s^2} - \frac{1}{2} e^{-6s} \mathcal{L}\{t\} = \boxed{\frac{1}{2s^2} - \frac{e^{-6s}}{2s^2}} \end{aligned}$$

$$(b) f(t) = (t^2 - 2t + 2)u_1(t)$$

$$\begin{aligned} f(t) &= [(t-1)^2 + 1] u_1(t) \\ &= (t-1)^2 u_1(t) + u_1(t) \\ \mathcal{L}\{(t-1)^2 u_1(t) + u_1(t)\} \\ &= e^{-s} \mathcal{L}\{t^2\} + \frac{e^{-s}}{s} \\ &= \boxed{e^{-s} \frac{2}{s^3} + \frac{e^{-s}}{s}} \end{aligned}$$

$$(c) f(t) = \int_0^t (t-\tau)^2 \cos 2\tau d\tau = (g * h)(t)$$

$$g(t-\tau) = (t-\tau)^2 \Rightarrow g(t) = t^2$$

$$h(\tau) = \cos 2\tau \Rightarrow h(t) = \cos 2t$$

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \mathcal{L}\{g(t)\} \cdot \mathcal{L}\{h(t)\} \\ &= \mathcal{L}\{t^2\} \cdot \mathcal{L}\{\cos 2t\} \\ &= \boxed{\frac{2}{s^3} \cdot \frac{s}{s^2+4}} \end{aligned}$$

$$(d) f(t) = t \cos 3t$$

$$\mathcal{L}\{t \cos 3t\} = (-1) \frac{d}{ds} \{ \mathcal{L}\{\cos 3t\} \}$$

$$= - \frac{d}{ds} \left(\frac{s}{s^2+9} \right)$$

$$= - \frac{s^2+9 \cdot 2s(s)}{(s^2+9)^2}$$

$$= \boxed{\frac{s^2-9}{(s^2+9)^2}}$$

$$(e) f(t) = e^t \delta(t-1)$$

$$\mathcal{L}\{\delta(t-1)\} = e^{-s}$$

$$\mathcal{L}\{e^t \delta(t-1)\} = \boxed{e^{-(s-1)}}$$

2. Find the inverse Laplace transform of the given function.

$$(a) F(s) = \frac{2s+6}{s^2-4s+8}$$

$$\begin{aligned} \frac{2s+6}{s^2-4s+8} &= \frac{2s+6}{(s-2)^2+4} = 2 \frac{s+3}{(s-2)^2+4} \\ &= 2 \frac{s-2+5}{(s-2)^2+4} = 2 \frac{s-2}{(s-2)^2+4} + 5 \cdot \frac{2}{(s-2)^2+4} \end{aligned}$$

$$\begin{aligned} \mathcal{L}^{-1}\{F(s)\} &= 2 \mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+4}\right\} + 5 \mathcal{L}^{-1}\left\{\frac{2}{(s-2)^2+4}\right\} \\ &= \boxed{2e^{2t} \cos 2t + 5e^{2t} \sin 2t} \end{aligned}$$

$$(b) F(s) = \frac{e^{-2s}}{s^2+s-2}$$

$$\begin{aligned} \frac{1}{s^2+s-2} &= \frac{1}{(s+2)(s-1)} = \frac{A}{s+2} + \frac{B}{s-1} \\ &= \frac{A(s-1)+B(s+2)}{(s+2)(s-1)} \end{aligned}$$

$$1 = A(s-1) + B(s+2)$$

$$s=1: 1 = 3B \Rightarrow B = 1/3 \quad |$$

$$s=-2: 1 = -3A \Rightarrow A = -1/3$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+s-2}\right\} = \mathcal{L}^{-1}\left\{-\frac{1}{3} \frac{1}{s+2} + \frac{1}{3} \frac{1}{s-1}\right\}$$

$$= -\frac{1}{3} e^{-2t} + \frac{1}{3} e^t$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2+s-2}\right\} = \boxed{\left[-\frac{1}{3} e^{-2(t-2)} + \frac{1}{3} e^{(t-2)}\right] u_2(t)}$$

3. Solve the initial value problem using the Laplace transform:

$$(a) y'' + 4y = \begin{cases} t, & 0 \leq t < 1 \\ 1, & t \geq 1 \end{cases}, y(0) = y'(0) = 0$$

$$g(t) = t + (1-t)u_1(t)$$

$$\begin{aligned} \mathcal{L}\{g(t)\} &= \frac{1}{s^2} - \mathcal{L}\{(t-1)u_1(t)\} \\ &= \frac{1}{s^2} - e^{-s} \mathcal{L}\{t\} \\ &= \frac{1}{s^2} - e^{-s} \frac{1}{s} \end{aligned}$$

$$\mathcal{L}\{y(t)\} = Y(s)$$

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0) = sY(s)$$

$$\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s)$$

$$\mathcal{L}\{y'' + 4y\} = \mathcal{L}\{g(t)\}$$

$$(s^2 + 4)Y(s) = \frac{1}{s^2} - \frac{e^{-s}}{s}$$

$$Y(s) = \frac{1}{s^2(s^2+4)} - \frac{e^{-s}}{s^2(s^2+4)}$$

Partial fractions:

$$\begin{aligned} \frac{1}{s^2(s^2+4)} &= \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+4} \\ &= \frac{As(s^2+4) + B(s^2+4) + (Cs+D)s^2}{s^2(s^2+4)} \end{aligned}$$

$$1 = s^3(A+C) + s^2(B+D) + s(4A) + 4B$$

$$s^3: D = A + C$$

$$s^2: 0 = B + D$$

$$s: 0 = 4A$$

$$1: 1 = 4B$$

$$A = C = 0$$

$$B = 1/4, D = -1/4$$

$$\frac{1}{s^2(s^2+4)} = \frac{1}{4} \frac{1}{s^2} - \frac{1}{4} \frac{1}{s^2+4}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+4)}\right\} = \frac{1}{4}t - \frac{1}{8}\sin 2t$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2(s^2+4)}\right\} = u_1(t) \left[\frac{1}{4}(t-1) - \frac{1}{8}\sin 2(t-1) \right]$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+4)} - \frac{e^{-s}}{s^2(s^2+4)}\right\}$$

$$= \left[\frac{1}{4}t - \frac{1}{8}\sin 2t + u_1(t) \left[\frac{1}{4}(t-1) - \frac{1}{8}\sin 2(t-1) \right] \right]$$

$$(b) y'' + 2y' + 3y = \delta(t - 3\pi), y(0) = y'(0) = 0$$

$$\mathcal{L}\{y'' + 2y' + 3y\} = \mathcal{L}\{\delta(t - 3\pi)\}$$

$$\mathcal{L}\{y\} = Y(s)$$

$$\mathcal{L}\{y'\} = sY(s) - y(0)$$

$$= sY(s)$$

$$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0)$$

$$= s^2Y(s)$$

$$\mathcal{L}\{\delta(t - 3\pi)\} = e^{-3\pi s}$$

$$(s^2 + 2s + 3)Y(s) = e^{-3\pi s}$$

$$Y(s) = \frac{1}{s^2 + 2s + 3} e^{-3\pi s}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{e^{-3\pi s}}{s^2 + 2s + 3}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2 + 2s + 3}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2 + 2}\right\}$$

$$= \frac{1}{\sqrt{2}} \mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{(s+1)^2 + 2}\right\}$$

$$= \frac{1}{\sqrt{2}} e^{-t} \sin \sqrt{2} t$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-3\pi s}}{s^2 + 2s + 3}\right\}$$

$$= \mathcal{U}_{3\pi}(t) \frac{1}{\sqrt{2}} e^{-(t-3\pi)} \sin \sqrt{2}(t-3\pi) = y(t)$$

$$(c) y'' + 4y' + 4y = g(t), y(0) = 2, y'(0) = -3$$

$$\mathcal{L}\{y'' + 4y' + 4y\} = \mathcal{L}\{g(t)\}$$

$$\mathcal{L}\{g(t)\} = G(s)$$

$$\mathcal{L}\{y\} = Y(s)$$

$$\mathcal{L}\{y'\} = sY(s) - y(0) = sY(s) - 2$$

$$\begin{aligned} \mathcal{L}\{y''\} &= s^2Y(s) - sy(0) - y'(0) \\ &= s^2Y(s) - 2s + 3 \end{aligned}$$

$$s^2Y(s) - 2s + 3 + 4sY(s) - 8 + 4Y(s) = G(s)$$

$$Y(s)(s^2 + 4s + 4) = G(s) + 2s + 5$$

$$Y(s) = \frac{G(s)}{s^2 + 4s + 4} + \frac{2s + 5}{s^2 + 4s + 4}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{G(s)}{s^2 + 4s + 4} + \frac{2s + 5}{s^2 + 4s + 4}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2 + 4s + 4}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\}$$

$$= e^{-2t} t$$

$$\frac{2s + 5}{s^2 + 4s + 4} = \frac{A}{s+2} + \frac{B}{(s+2)^2}$$

$$= \frac{A(s+2) + B}{(s+2)^2}$$

$$2s + 5 = A(s+2) + B$$

$$s = -2: 1 = B$$

$$s = 0: 5 = 2A + B, 2A = 5 - B = 4$$

$$A = 2$$

$$\frac{2s + 5}{s^2 + 4s + 4} = \frac{2}{s+2} + \frac{1}{(s+2)^2}$$

$$\mathcal{L}^{-1}\left\{\frac{2s + 5}{s^2 + 4s + 4}\right\} = 2e^{-2t} + e^{-2t} t$$

$$y(t) = \int_0^t g(t-\tau) e^{-2\tau} \tau d\tau + 2e^{-2t} + e^{-2t} t$$

4. Find A^{-1} if $A = \begin{pmatrix} 1+i & -1+2i \\ 3+2i & 2-i \end{pmatrix}$

$$\det A = (1+i)(2-i) - (-1+2i)(3+2i) = 2-i+2i-i^2 + 3+2i-6i-4i^2$$

$$= 5+5-3i = 10-3i$$

$$\frac{1}{\det A} = \frac{1}{10-3i} = \frac{10+3i}{(10-3i)(10+3i)} = \frac{10+3i}{100-9i^2} = \frac{10+3i}{109}$$

$$A^{-1} = \frac{10+3i}{109} \begin{pmatrix} 2-i & 1-2i \\ -3-2i & 1+i \end{pmatrix}$$

5. Find BA if $A = \begin{pmatrix} 1+i & -1+2i \\ 3+2i & 2-i \end{pmatrix}$, $B = \begin{pmatrix} i & 3 \\ 2 & -2i \end{pmatrix}$

$$BA = \begin{pmatrix} i & 3 \\ 2 & -2i \end{pmatrix} \begin{pmatrix} 1+i & -1+2i \\ 3+2i & 2-i \end{pmatrix} = \begin{pmatrix} i(1+i)+3(3+2i) & i(-1+2i)+3(2-i) \\ 2(1+i)-2i(3+2i) & 2(-1+2i)-2i(2-i) \end{pmatrix}$$

$$= \begin{pmatrix} i+i^2+9+6i & -i+2i^2+6-3i \\ 2+2i-6i-4i^2 & -2+4i-4i+4i^2 \end{pmatrix} = \begin{pmatrix} 8+7i & 4-4i \\ 6-4i & -6 \end{pmatrix}$$

6. Find the general solution of the system

$$(a) \mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x} \quad A = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}, A - \lambda I = \begin{pmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{pmatrix}$$

Eigenvalues: $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix} = (1-\lambda)(-2-\lambda) - 4 = 0$$
$$-2 - \lambda + 2\lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 + \lambda - 6 = 0$$

$$(\lambda + 3)(\lambda - 2) = 0$$

$\lambda_1 = -3, \lambda_2 = 2$ - eigenvalues

Corresponding eigenvectors:

$\lambda_1 = -3$. Corresponding eigenvector $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is a solution of the system $(A + 3I)\vec{v} = \vec{0}$

$$\begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$4v_1 + v_2 = 0 \Rightarrow v_2 = -4v_1$$

$$\vec{v} = \begin{pmatrix} v_1 \\ -4v_1 \end{pmatrix} \stackrel{v_1=1}{=} \boxed{\begin{pmatrix} 1 \\ -4 \end{pmatrix} \text{ corresponds to } \lambda_1 = -3}$$

$$\lambda_2 = 2. \vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

$$(A - 2I)\vec{w} = \vec{0}$$

$$\begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-w_1 + w_2 = 0 \Rightarrow w_1 = w_2$$

$$\vec{w} = \begin{pmatrix} w_1 \\ w_1 \end{pmatrix} \stackrel{w_1=1}{=} \boxed{\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ corresponds to } \lambda_2 = 2}$$

General solution:

$$\boxed{\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}}$$

$$(b) \mathbf{x}' = \begin{pmatrix} -3 & -1 \\ 1 & -1 \end{pmatrix} \mathbf{x}$$

$$A = \begin{pmatrix} -3 & -1 \\ 1 & -1 \end{pmatrix}, A - \lambda I = \begin{pmatrix} -3-\lambda & -1 \\ 1 & -1-\lambda \end{pmatrix}$$

Eigenvalues: $\det(A - \lambda I) = 0$

$$\begin{vmatrix} -3-\lambda & -1 \\ 1 & -1-\lambda \end{vmatrix} = (-3-\lambda)(-1-\lambda) + 1$$

$$= 3 + 3\lambda + \lambda + \lambda^2 + 1$$

$$= \lambda^2 + 4\lambda + 4 = 0$$

$$(\lambda + 2)^2 = 0$$

$\lambda = -2$ - repeated eigenvalue

Corresponding eigenvector:

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$(A + 2I)\vec{v} = \vec{0}$$

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$v_1 + v_2 = 0 \Rightarrow v_1 = -v_2$$

$$\vec{v} = \begin{pmatrix} v_1 \\ -v_1 \end{pmatrix} \stackrel{v_1=1}{=} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\text{General solution } \vec{x}(t) = c_1 t e^{-2t} \vec{v} + c_2 e^{-2t} \vec{y}$$

$\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ and \vec{y} is a solution of

$$(A + 2I)\vec{y} = \vec{b}$$

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$-y_1 - y_2 = 1 \Rightarrow y_2 = -y_1 + 1$$

$$\vec{y} = \begin{pmatrix} y_1 \\ 1 - y_1 \end{pmatrix} \stackrel{y_1=0}{=} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

General solution:

$$\boxed{\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^{-2t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t}}$$

$$(c) \mathbf{x}' = \begin{pmatrix} 3 & 9 \\ 1 & -3 \end{pmatrix} \mathbf{x}$$

$$\mathbf{x}' = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix} \mathbf{x}, \mathbf{A} = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix}, \mathbf{A} - \lambda \mathbf{I} = \begin{pmatrix} -3-\lambda & 2 \\ -1 & -1-\lambda \end{pmatrix}$$

eigenvalues:

$$\begin{vmatrix} -3-\lambda & 2 \\ -1 & -1-\lambda \end{vmatrix} = (3+\lambda)(1+\lambda) + 2$$

$$= 3 + 4\lambda + \lambda^2 + 2 = 0$$

$$\lambda^2 + 4\lambda + 5 = 0$$

$$\lambda_1 = \frac{-4 + \sqrt{16 - 20}}{2} = -2 + i, \lambda_2 = -2 - i$$

Corresponding eigenvector: $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$,

$$(\mathbf{A} - (-2+i)\mathbf{I})\vec{v} = \vec{0}$$

$$\begin{pmatrix} -3 - (-2+i) & 2 \\ -1 & -1 - (-2+i) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1-i & 2 \\ -1 & 1-i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -(1+i)v_1 + 2v_2 = 0 \\ -v_1 + (1-i)v_2 = 0 \end{cases} \Rightarrow v_1 = (1-i)v_2$$

$$\vec{v} = \begin{pmatrix} (1-i)v_2 \\ v_2 \end{pmatrix} \stackrel{v_2=1}{=} \begin{pmatrix} 1-i \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

solutions:

$$\vec{v} e^{\lambda_1 t} = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right] e^{(-2+i)t}$$

$$= \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right] e^{-2t} (\cos t + i \sin t)$$

$$= e^{-2t} \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos t + i \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin t + i \begin{pmatrix} -1 \\ 0 \end{pmatrix} \cos t + i^2 \begin{pmatrix} -1 \\ 0 \end{pmatrix} \sin t \right]$$

$$= e^{-2t} \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos t - \begin{pmatrix} -1 \\ 0 \end{pmatrix} \sin t + i \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin t + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \cos t \right] \right\}$$

$$= e^{-2t} \left[\begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix} + i \begin{pmatrix} \sin t - \cos t \\ \sin t \end{pmatrix} \right]$$

General solution $\boxed{\mathbf{x}(t) = \left[c_1 \begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix} + c_2 \begin{pmatrix} \sin t - \cos t \\ \sin t \end{pmatrix} \right] e^{-2t}}$

