MATH 308-518,519,520 Sample problems for the Final Spring 2014

1. A large tank initially contains 10 L of fresh water. A brine containing 20 g/L of salt flows into the tank at a rate of 3 L/min. The solution inside the tank is kept well stirred and flows out of the tank at the rate 2 L/min. Determine the concentration of salt in the tank as a function of time.

2. Suppose that a sum S_0 is invested at an annual rate of return r compounded continuously.

   (a) Find the time T required for the original sum to double in value as a function of r.

   (b) Determine T if $r = 7\%$.

   (c) Find the return rate that must be achieved if the initial investment is to double in 8 years.

3. An object with temperature 150^0 is placed in a freezer whose temperature is 30^0. Assume that the temperature of the freezer remains essentially constant.

   (a) If the object is cooled to 120^0 after 8 min, what will its temperature be after 18 min?

   (b) When will its temperature be 60^0?

4. Determine (without solving the problem) an interval in which the solution to the initial value problem
$$(4-t^2)y' + 2ty = 3t^2, \quad y(1) = -3$$
is certain to exist.

5. Solve the initial value problem
$$y' = \frac{t^2}{1+t^3}, \quad y(0) = y_0$$
and determine how the interval in which the solution exists depends on the initial value y_0.

6. Solve the following initial value problem
$$\sqrt{y}dt + (1+t)dy = 0 \quad y(0) = 1.$$

7. Find the general solution to the equation
$$(t^2-1)y' + 2ty + 3 = 0$$

8. Solve the initial value problem
$$(ye^{xy}\cos(2x) - 2e^{xy}\sin(2x) + 2x)dx + (xe^{xy}\cos(2x) - 3)dy = 0, \quad y(0) = -1$$

9. Find an integrating factor for the equation
$$(3xy + y^2) + (x^2 + xy)y' = 0$$
and then solve the equation.

10. Solve the initial value problem

$$6y'' - 5y' + y = 0, \qquad y(0) = 4, y'(0) = 0$$

11. Find the general solution to the equation

$$4y'' - 12y' + 9y = 0$$

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12. The Existence and Uniqueness Theorem guarantees that the solution to

$$x^3y'' + \frac{x}{\sin x}y' - \frac{2}{x-5}\,y = 0, \quad y(2) = 6, \quad y'(2) = 7$$

uniquely exists on

(a) $(-\pi,\, \pi)$
(b) $(0,\, \pi)$
(c) $(5,\, \infty)$
(d) $(0,\, 5)$

13. All of the following pairs of functions form a fundamental set of solutions to some second order differential equation on $(-\infty,\, \infty)$ EXCEPT

(a) $1, \quad e^{-t}$
(b) $\cos t, \quad \sin(t + 2\pi)$
(c) $e^{-2t}\cos 2t, \quad e^{-2t}\sin 2t$
(d) $e^{5t}, \quad e^{5t-1}$

14. Which of the following will be a particular solution to the equation

$$4y'' + 4y' + y = 24x\mathrm{e}^{\frac{x}{2}}?$$

(a) $x^2(Ax + B)\mathrm{e}^{\frac{x}{2}}$
(b) $(Ax + B)\mathrm{e}^{\frac{x}{2}}$
(c) $x(Ax + B)\mathrm{e}^{\frac{x}{2}}$
(d) $(Ax + B)\sin\frac{x}{2} + (Cx + D)\cos\frac{x}{2}$

15. A 2-kg mass is attached to a spring with stiffness $k = 50$ N/m. The damping force is negligible. What is the resonance frequency for the system?

(a) 5
(b) 2
(c) 3
(d) 4

16. The motion of the mass-spring system with damping is governed by $y'' + 2y' + y = 0, \quad y(0) = 1, y'(0) = -3$. This motion is

(a) undamped

(b) underdamped

(c) critically damped

(d) overdamped

17. $e^{2+\frac{3\pi}{4}i} =$

(a) π

(b) $\frac{\sqrt{2}}{2}(1+i)e^2$

(c) $\frac{\sqrt{2}}{2}(1-i)e^2$

(d) $\frac{\sqrt{2}}{2}(-1+i)e^2$

18. A 2-kg mass is attached to a spring with stiffness $k = 50$ N/m. The mass is displaced 1/4 m to the left of the equilibrium point and given a velocity of 1 m/sec to the left. The damping force is negligible. The amplitude of this vibration is

(a) $\frac{\sqrt{41}}{20}$

(b) 1

(c) $\frac{\sqrt{20}}{41}$

(d) $\frac{1}{4}$

19. The FSS to the equation $y'' - 2y' + 5y = 0$ is

(a) $\{\cos x, \sin x\}$

(b) $\{e^x \cos 2x, e^x \sin 2x\}$

(c) $\{e^x, xe^x\}$

(d) $\{e^x, e^{-x}\}$

20. Given that $y_1(x) = -\frac{1}{2}x^2 + \frac{1}{2}x - \frac{3}{4}$ is a solution to $y'' - y' - 2y = x^2$ and $y_2(x) = \frac{1}{4}e^{3x}$ is a solution to $y'' - y' - 2y = e^{3x}$. A solution to $y'' - y' - 2y = 2x^2 - e^{3x}$ is

(a) $-x^2 + x - \frac{3}{2} - \frac{1}{4}e^{3x}$
(b) $x^2 - x - \frac{3}{2} - \frac{1}{4}e^{3x}$
(c) $-x^2 + x + \frac{3}{4} - e^{3x}$
(d) $x - \frac{3}{2} - \frac{1}{4}e^{3x}$

21. The Wronskian of two functions $y_1(x) = x + 2x^2$ an $y_2(x) = 2^x$ is

(a) $2^x(1 + 4x - x(1 + 2x))$
(b) $-2^x(1 + 4x - x\ln 2(1 + 2x))$
(c) $(1 + 4x - x(1 + 2x))$
(d) $2^x(1 + 2x - x\ln 2(1 + 4x))$

22. Find a general solution to the equation

$$y'' + 6y' + 9y = \frac{e^{-3x}}{1 + 2x}$$

23. Find a general solution to the equation

$$4y'' + y' = 4x^3 + 48x^2 + 1$$

24. Given that $y_1(x) = x$ is a solution to

$$x^2y'' + xy' - y = 0,$$

find a second solution of this equation on $(0, +\infty)$.

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25. Find the Laplace transform of the given function.

(a) $f(t) = \begin{cases} \frac{t}{2}, & 0 \le t < 6 \\ 3, & t \ge 6 \end{cases}$
(b) $f(t) = (t^2 - 2t + 2)u_1(t)$
(c) $f(t) = \int_0^t (t - \tau)^2 \cos 2\tau d\tau$
(d) $f(t) = t\cos 3t$
(e) $f(t) = e^t\delta(t - 1)$

26. Find the inverse Laplace transform of the given function.

(a) $F(s) = \dfrac{2s+6}{s^2-4s+8}$

(b) $F(s) = \dfrac{e^{-2s}}{s^2+s-2}$

27. Solve the initial value problem using the Laplace transform:

(a) $y'' + 4y = \begin{cases} t, & 0 \le t < 1 \\ 1, & t \ge 1 \end{cases}$, $y(0) = y'(0) = 0$

(b) $y'' + 2y' + 3y = \delta(t - 3\pi)$, $y(0) = y'(0) = 0$

(c) $y'' + 4y' + 4y = g(t)$, $y(0) = 2$, $y'(0) = -3$

28. Find A^{-1} if $A = \begin{pmatrix} 1+i & -1+2i \\ 3+2i & 2-i \end{pmatrix}$

29. Find BA if $A = \begin{pmatrix} 1+i & -1+2i \\ 3+2i & 2-i \end{pmatrix}$, $B = \begin{pmatrix} i & 3 \\ 2 & -2i \end{pmatrix}$

30. Find the general solution of the system

(a) $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x}$

(b) $\mathbf{x}' = \begin{pmatrix} -3 & -1 \\ 1 & -1 \end{pmatrix} \mathbf{x}$

(c) $\mathbf{x}' = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix} \mathbf{x}$