S.1.1. Some basic math models. Direction field.

- Equation containing derivatives of an unknown function is called a <u>differential equation</u>.

<u>Examples.</u> 1) $y'' + y^2 y' + \sin x = 0$ $\quad y = y(x)$ $\quad y' = \frac{dy}{dx}$

x is the independent variable
y is the dependent variable

2) $t^2 y'' + ty = 0$ $\quad y = y(t)$ $\quad y' = \frac{dy}{dt}$

t is the independent variable
y is the dependent variable.

- a differential equation that describes some physical process is often called a <u>mathematical model</u> of the process.

<u>Examples.</u> 1) A Falling object.

1a) neglect the air resistance. $\quad \downarrow m\vec{g} \quad \downarrow \vec{v}$

h is the height of the object at time t.

$$\vec{F} = m\vec{g}$$

2nd Newton's law $\quad \vec{F} = m\vec{a}$

$$\cancel{m}\vec{g} = \cancel{m}\vec{a}$$

$$\vec{a} = \vec{g}$$

a is the acceleration of the object at time t.

$$a = a(t)$$

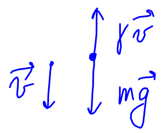
$$a = \frac{d^2h}{dt^2}$$

$$\boxed{\frac{d^2h}{dt^2} = g = 9.81}$$

1B) air resistance is proportional to the velocity of the object.

γ is the drag coefficient $(\gamma < 0)$

$\vec{F} = \gamma\vec{v} + m\vec{g}$



v is the velocity of the object at time t.

$m\vec{a} = \gamma\vec{v} + m\vec{g}$

$ma = mg - \gamma v$, $a = \frac{dv}{dt}$

$\boxed{m\frac{dv}{dt} = mg - \gamma v}$ $(m, g, \gamma$ are constants$)$.

2) <u>Field mice and Owls.</u>

Consider a population of field mice who inhabit a certain rural area. In the absence of predators the rate of change of the mouse population is proportional to the current population

$p(t)$ is the population of mice at time t

$\frac{dp}{dt} = rp$, where r is the growth rate.

If we assume that predator rate is a constant k $(k > 0)$, then

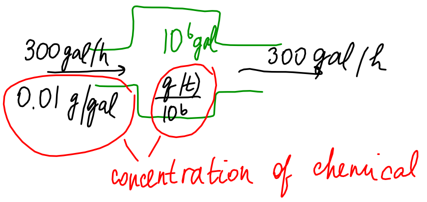
$$\boxed{\frac{dp}{dt} = rp - k}$$

Constructing math models:

1) Identify the independent and dependent variables and assign letters to represent them.
2) Choose the units of measurements for each variable.
3) Articulate the basic principle that underlines or governs the problem you are investigating.
4) Express the principle or law from step 3 in terms of the variables from step 1.
5) Make sure that each term in the equation has the same physical units.
6) The result of step 4 is a single differential equation which constitutes the math. model.

Example. A pond initially contains 10^6 gal of water and an unknown amount of an undesirable chemical. Water containing 0.01 g of this chemical per gallon flows into the pond at a rate of 300 gal/h. The mixture flows out at the same rate. Assume that the chemical is uniformly distributed throughout the pond. Write a differential equation for the amount of chemical in the pond at any time.

$q(t)$ is the amount of chemical in the pond at any time.



$\boxed{\text{rate of change of the amount of chemical in the pond}} = \boxed{\text{input rate}} - \boxed{\text{output rate}}$

rate of change $= \frac{dq}{dt}$

input rate $= (0.01)(300) = 3\ (g/h)$

output rate $= \frac{q}{10^6}(300) = 3q \times 10^{-4}\ (g/h)$

$$\boxed{\frac{dq}{dt} = 3 - 3q \times 10^{-4}}$$

Section 1.2. Solutions of some differential equations.

• An order of a differential equation is the order of the highest derivative present in the equation.

Examples. 1) $y\,y''' - 10y = \sin t$ of order 3

2) $m\frac{dv}{dt} = mg - \gamma v$ of order 1

a general form of a differential equation of order n is

$$\boxed{F(t, y, y', y'', \dots, y^{(n)}) = 0} \quad (y = y(t))$$

Sometime we can solve this equation for $y^{(n)}$

$$\boxed{y^{(n)} = f(t, y, y', \dots, y^{(n-1)})}$$

Definition. A function $g(t)$ is called a solution to a diff. eq. of order n if it satisfies the equation for all t in the interval I.

Each differential equation has infinitely many solutions.

The graph of $g(t)$ (for some C) is called the integral curve.

Example. Determine whether a function is a solution to the given differential equation:

1) $g(t) = 5t^2$

$tg' = 2g$

$g'(t) = 10t$

Plug g and g' into the equation:

$$t\underbrace{(10t)}_{g'} \overset{?}{=} 2\underbrace{(5t^2)}_{g}$$

$\boxed{\text{YES, for all } -\infty < t < \infty}$

$10t^2 = 10t^2$ for all t from $-\infty$ to $+\infty$

2) $y(t) = \sqrt{2t - t^2} = (2t - t^2)^{1/2}$

$y^2 y'' + 1 = 0$

$$y'(t) = \frac{1}{2}(2t - t^2)^{-1/2}(2 - 2t)$$

$$= (1 - t)(2t - t^2)^{-1/2}$$

$$y''(t) = -(2t - t^2)^{-1/2} + (1 - t)\left(-\frac{1}{2}\right)(2t - t^2)^{-3/2}(2 - 2t)$$

$$= -(2t - t^2)^{-1/2} - (1 - t)^2(2t - t^2)^{-3/2}$$

$$= -\frac{1}{\sqrt{2t - t^2}} - \frac{(1 - t)^2}{(2t - t^2)\sqrt{2t - t^2}}$$

$$= -\frac{2t - t^2 + (1 - t)^2}{(2t - t^2)\sqrt{2t - t^2}}$$

$$= -\frac{\cancel{2t} - \cancel{t^2} + 1 - \cancel{2t} + \cancel{t^2}}{(2t - t^2)\sqrt{2t - t^2}}$$

$$= -\frac{1}{(2t - t^2)^{3/2}}$$

$$y^2 y'' + 1 = 0$$

$$\cancel{(2t - t^2)}\left(-\frac{1}{\cancel{(2t - t^2)}\sqrt{2t - t^2}}\right) + 1 \overset{?}{=} 0$$

$$\underbrace{1 - \frac{1}{\sqrt{2t - t^2}}}_{\text{"0 only if } t = 1.} \overset{?}{=} 0$$

$\boxed{\text{YES, @ } t = 1 \text{ only}}$

Section 1.3. Classification of differential equations.

$$y^{(n)} = f(t, y, y', \dots, y^{(n-1)})$$

has infinitely many solutions.

Sometimes we need to find the solution of the equation that satisfies the conditions

$$y(t_0) = y_0, \; y'(t_0) = y_1, \dots$$

(or find the integral curve that passes through (t_0, y_0))

Solve the initial value problem (IVP)

$$y^{(n)} = f(t, y, y', \dots, y^{(n-1)})$$

$$y(t_0) = y_0$$

$$y'(t_0) = y_1$$

$$\dots\dots$$

For the first order differential equation the IVP is formulated in the following way:

Find the solution of the equation

$$y' = f(x, y)$$

that satisfies the condition $y(t_0) = y_0$

• The differential equation of the form

$$a_n(t)\, y^{(n)}(t) + a_{n-1}(t)\, y^{(n-1)}(t) + \dots + a_1(t)\, y(t) + a_0(t) = 0$$

is called the linear differential equation

If the equation is not linear, then it is nonlinear.