1. Find an integrating factor and then solve the equation.

   (a) $(3x^2 + y)dx + (x^2y - x)dy = 0$

   (b) $(2y^2 + 2y + 4x^2)dx + (2xy + x)dy = 0$

#1 a)

$M(x,y) = 3x^2 + y$

$N(x,y) = x^2y - x$

$$\frac{M_y - N_x}{N(x,y)} = \frac{1-(2xy-1)}{x^2y - x} = \frac{2-2xy}{x(xy-1)} = -\frac{2}{x}$$

$$\mu(x): \frac{d\mu}{dx} = -\frac{2\mu}{x}$$

#1 b) $(2y^2+2y+4x^2)dx + (2xy+x)dy = 0$

$M(x,y) = 2y^2 + 2y + 4x^2$

$N(x,y) = 2xy + x$

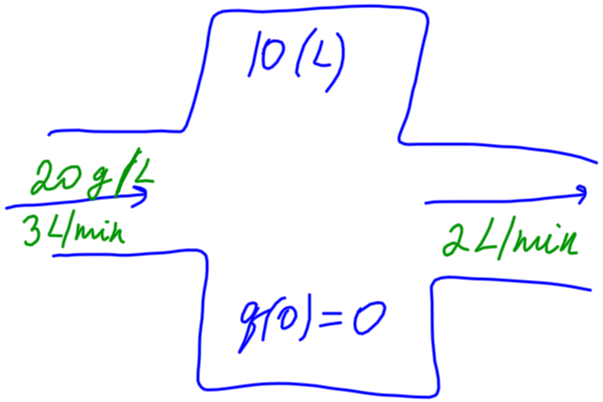
$$\frac{M_y - N_x}{N(x,y)} = \frac{4y+2-(2y+1)}{2xy+x} = \frac{2y+1}{x(2y+1)} = \frac{1}{x}$$

$$\mu(x): \frac{d\mu}{dx} = \frac{\mu}{x}$$

MATH 308-518,519,520 Spring 2013 Practice Test I

over sections 1.1–1.3, 2.1–2.6, 3.1

1. A large tank initially contains 10 L of fresh water. A brine containing 20 g/L of salt flows into the tank at a rate of 3 L/min. The solution inside the tank is kept well stirred and flows out of the tank at ~~the~~ a rate of 2 L/min. Determine the concentration of salt in the tank as a function of time.



$q(t)$ is the mass of salt in the tank at time t.

$$\frac{dq}{dt} = \text{rate in} - \text{rate out}$$

$$\text{rate in} = 3(20)$$

$$\text{rate out} = \frac{q(t)}{10+(3-2)t} \cdot 2$$

$$\frac{dq}{dt} = 60 - \frac{2q(t)}{10+t} \quad \text{IVP}$$

$$q(0)=0$$

Solve for $q(t)$.

So $\frac{q(t)}{10+t}$ concentration

2. Suppose that a sum S_0 is invested at an annual rate of return r compounded continuously.

(a) Find the time T required for the original sum to double in value as a function of r.

$S(t)$ is the balance at time t.

$$\frac{dS}{dt} = rS$$

$$S(0) = S_0$$

Find T such that $S(T) = 2S_0$

$$\frac{dS}{S} = r\,dt$$

$$\ln|S| = rt + c$$

$$S(t) = Ce^{rt}$$

$$S_0 = S(0) = C$$

$$\boxed{S(t) = S_0 e^{rt}}$$

$$S(T) = \cancel{S_0} e^{rT} = 2\cancel{S_0}$$

$$\boxed{T = \frac{\ln 2}{r}}$$

(b) Determine T if $r = 7\%$. $= 0.07$

$$T = \frac{\ln 2}{0.07} \approx \boxed{10 \text{ (years)}}$$

(c) Find the return rate that must be achieved if the initial investment is to double in 8 years.

$$8 = \frac{\ln 2}{r}$$

$$r = \frac{\ln 2}{8} \approx 0.09$$

$$\boxed{9\%}$$

3. An object with temperature 150^0 is placed in a freezer whose temperature is 30^0. Assume that the temperature of the freezer remains essentially constant.

(a) If the object is cooled to 120^0 after 8 min, what will its temperature be after 18 min?

$T(t)$ – temperature at time t

$$\frac{dT}{dt} = k(30 - T) \qquad T(0) = 150$$

$$T(8) = 120$$

Find $T(18) = ?$

$$\frac{dT}{dt} = -k(T-30)$$

$$\frac{dT}{T-30} = -k\,dt$$

$$\ln|T-30| = -kt + C$$

$$T - 30 = Ce^{-kt}$$

$$T = 30 + Ce^{-kt}$$

$$30 + C = 150$$

$$C = 120$$

$$T(t) = 30 + 120e^{-kt}$$

solve for k

$$T(8) = 30 + 120e^{-8k} = 120$$

$$e^{-8k} = \frac{3}{4}$$

$$k = -\frac{1}{8}\ln\left(\frac{3}{4}\right) \approx 0.036$$

$$T(t) = 30 + 120e^{-0.036t}$$

$$T(18) = \ldots$$

(b) When will its temperature be 60^0?

find t such that $T(t)=60$

4. Determine (without solving the problem) an interval in which the solution to the initial value problem

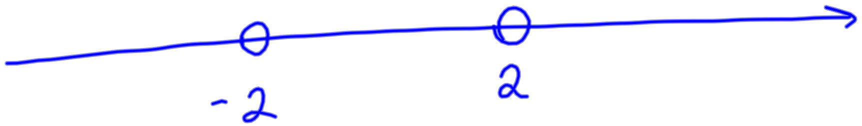
$$(4-t^2)y' + 2ty = 3t^2, \quad y(1) = -3$$

is certain to exist.

$$\boxed{(-2, 2)}$$

$$y' + \frac{2t}{4-t^2}\, y = \frac{3t^2}{4-t^2}$$

continuous if $t \neq \pm 2$



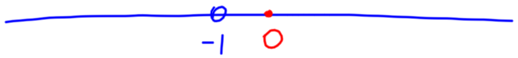
5. Solve the initial value problem

$$y' = \frac{t^2}{1+t^3}, \quad y(0) = y_0$$

and determine how the interval in which the solution exists depends on the initial value y_0.

$f(t,y) = \frac{t^2}{1+t^3}$ continuous if $t \neq -1$

$\frac{\partial f}{\partial y} = 0$



$y' = \frac{t^2}{1+t^3}, \quad y(t_0) = y_0$

if $t_0 > -1$, $(-1, \infty)$

if $t_0 < -1$, $(-\infty, -1)$

$0 > -1$, so $(-1, \infty)$ does not depend on y_0

Solve the initial value problem:

$\frac{dy}{dt} = \frac{t^2}{1+t^3}$

$dy = \frac{t^2}{1+t^3}\, dt$

$y(t) = \frac{1}{3} \ln|1+t^3| + C$

$y(0) = C = y_0$

$y(t) = \frac{1}{3} \ln|1+t^3| + y_0$ exists everywhere except $t = -1$.

6. Solve the following initial value problem

$$\sqrt{y}dt + (1+t)dy = 0 \quad y(0) = 1.$$

7. Find the general solution to the equation

$$(t^2-1)y' + 2ty + 3 = 0$$

$$y' + \frac{2t}{t^2-1}\, y = -\frac{3}{t^2-1}$$

$$p(t) = \frac{2t}{t^2-1}\ ; \quad g(t) = -\frac{3}{t^2-1}$$

Integrating factor:

$$\frac{d\mu(t)}{dt} = +\frac{2t\mu}{t^2-1}$$

$$\frac{d\mu}{\mu} = \frac{2t}{t^2-1}\, dt$$

$$\mu(t) = t^2-1$$

$$(t^2-1)\, y(t) = \int -\frac{3}{t^2-1}(t^2-1)\, dt$$

$$= -3t + C$$

$$\boxed{y(t) = -\frac{3t}{t^2-1} + \frac{C}{t^2-1}}$$

8. Solve the initial value problem

$$(uvw)' = u'vw + uv'w + uvw'$$

$$\underbrace{(ye^{xy}\cos(2x) - 2e^{xy}\sin(2x) + 2x)}_{M(x,y)}dx + \underbrace{(xe^{xy}\cos(2x) - 3)}_{N(x,y)}dy = 0, \quad y(0) = -1$$

$$M_y = e^{xy}\cos(2x) + xye^{xy}\cos(2x) - 2xe^{xy}\sin(2x)$$

$$N_x = e^{xy}\cos(2x) + xye^{xy}\cos(2x) - 2\sin(2x)xe^{xy}$$

Exact

$$\begin{cases} \dfrac{\partial F}{\partial x} = ye^{xy}\cos(2x) - 2e^{xy}\sin(2x) + 2x \\ \displaystyle\int \dfrac{\partial F}{\partial y}dy = \int (xe^{xy}\cos(2x) - 3)\,dy \end{cases}$$

$$\boxed{\int e^{xy}dy = \frac{1}{x}e^{xy} + c}$$

$$F(x,y) = x\frac{1}{x}e^{xy}\cos(2x) - 3y + h(x)$$

$$= e^{xy}\cos(2x) - 3y + h(x)$$

$$\frac{\partial F}{\partial x} = ye^{xy}\cos(2x) - 2e^{xy}\sin(2x) + h'(x)$$

$h'(x) = 2x$ or $h(x) = x^2 + C$

$$F(x,y) = e^{xy}\cos(2x) - 3y + x^2 + C$$

General solution: $\boxed{e^{xy}\cos(2x) - 3y + x^2 + C = 0}$

$y(0) = -1$. Plug $y=-1$ and $x=0$ into the general solution:

$$e^{(0)}\cos(0) - 3(-1) + 0 + C = 0$$

$$C = -4$$

$$\boxed{e^{xy}\cos(2x) - 3y + x^2 - 4 = 0}$$

9. Find an integrating factor for the equation

$(3xy+y^2)dx+(x^2+xy)\,dy=0$

$$\underbrace{(3xy+y^2)}_{M(x,y)}+\underbrace{(x^2+xy)}_{N(x,y)}y'=0$$

and then solve the equation.

$$\frac{M_y-N_x}{N(x,y)}=\frac{3x+2y-(2x+y)}{x^2+xy}$$

$$=\frac{x+y}{x(x+y)}$$

$$=\frac{1}{x}\ \text{(depends on } x \text{ only)}$$

Integrating factor:

$$\frac{d\mu}{dx}=\frac{\mu}{x}$$

$$\frac{d\mu}{\mu}=\frac{dx}{x}$$

$$\mu(x)=x$$

$$\underbrace{(3x^2y+xy^2)}_{M(x,y)}dx+\underbrace{(x^3+x^2y)}_{N(x,y)}dy=0$$

$$M_y=3x^2+2xy$$
$$\parallel$$
$$N_x=3x^2+2xy$$

Exact

$$\begin{cases}\dfrac{\partial F}{\partial x}=3x^2y+xy^2\\ \displaystyle\int\frac{\partial F}{\partial y}dy=\int x^3+x^2y\,dy\end{cases}$$

$$F(x,y)=x^3y+\frac{x^2y^2}{2}+h(x)$$

$$\frac{\partial F}{\partial x}=3x^2y+xy^2+h'(x)$$

$$h'(x)=0 \quad\text{or}\quad h(x)=C$$

$$F(x,y)=x^3y+\frac{x^2y^2}{2}+C$$

General solution: $\boxed{x^3y+\frac{x^2y^2}{2}+C=0}$ $\boxed{y=0}$

10. Solve the initial value problem

$$6y'' - 5y' + y = 0, \qquad y(0) = 4, y'(0) = 0$$

11. Find the general solution to the equation

$$4y'' - 12y' + 9y = 0$$

$4r^2 - 12r + 9 = 0$

$(2r-3)^2 = 0$

$r = \frac{3}{2}$ repeated root

General solution:

$$\boxed{y(t) = (c_1 + c_2 t)e^{\frac{3}{2}t}}$$

W/R 1

2. Given the differential equation

$$\frac{dy}{dt} = ty - 1.$$

(a) What is the slope of the graph of the solutions at $(0, 1)$, at the point $(1, 1)$, at the point $(3, -1)$, at the point $(0, 0)$?

(b) Find all the points where the tangents to the solution curves are horizontal?

#2a

slope @ $(t, y) = ty - 1$

@ $(0,1) = -1$

@ $(1,1) = 0$

@ $(3,-1) = -4$

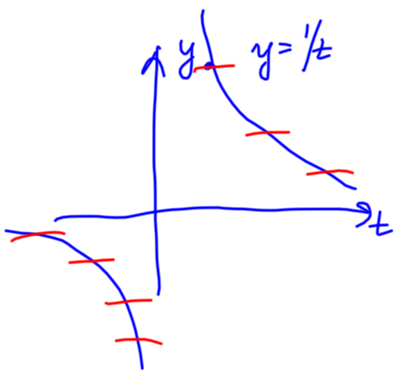
@ $(0,0) = -1$

#2b

slope $= 0$

$ty - 1 = 0$

$y = \frac{1}{t}$



The slope is horizontal at all points on the hyperbola $y = \frac{1}{t}$

HW 3 #3

Given the differential equation

$$\frac{dy}{dt} = 7y - y^2 - 10$$

(a) Find the equilibrium solutions.

(b) Sketch the phase line and determine whether the equilibrium solutions are stable, unstable, or semistable.

3a) $7y - y^2 - 10 = 0$

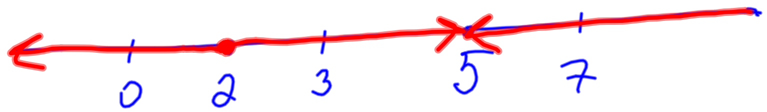
$y^2 - 7y + 10 = 0$

$(y-5)(y-2) = 0$

$y_1 = 2$ unstable

$y_2 = 5$ stable

3b)



$f(0) = -10 < 0$

$f(3) = 21 - 9 - 10 > 0$

$f(7) = 49 - 49 - 10 < 0$