Chapter 2. First Order Differential Equations

Section 2.2 **Separable Equations**

$$\boxed{\frac{dy}{dx} = f(x,y)}$$

$f(x,y) = g(x)h(y)$

Sometimes a function $f(x,y)$ can be represented as a product of two functions, one of which depens ONLY on x, another depens ONLY on y, or $f(x,y) = g(x)h(y)$. Then

$$\boxed{\frac{dy}{dx} = g(x)h(y).}$$

Definition. A differential equation $y' = f(x,y)$ is called separable if it can be written in the form

$$\boxed{M(x)dx + N(y)dy = 0}$$

Example 1. Determine whether the given equation is separable.

1. $(t - 2y)^2 y' = 2$

$(t-2y)^2$ cannot separate t and y.

$\boxed{\text{NO}}$

2. $y^4 e^y + (t^3+1)y' = y'(t^3+1)e^{2y}$

$(t^3+1)y' - y'(t^3+1)e^{2y} = -y^4 e^y$

$(t^3+1)y'(1-e^{2y}) = -y^4 e^y$

$y' = -\dfrac{y^4 e^y}{(t^3+1)(1-e^{2y})}$

$\boxed{\text{YES}}$

3. $yx \ln x dx - \sqrt{y} dy + x \ln x dx = 0$

$x \ln x dx (y+1) - \sqrt{y} dy$

$x \ln x dx - \dfrac{\sqrt{y}}{y+1} dy = 0$

$\boxed{\text{YES}}$

4. $y' = \cot^2\left(\frac{x}{2} + y - 1\right) + \frac{1}{2}$

$\boxed{\text{NO}}$

How to solve a separable equation?

$\frac{dy}{dx} = g(x)\,h(y)$

$y' = g(x)\,h(y)$, $y = y(x)$

$y' = \frac{dy}{dx}$

1. separate variables

$\frac{dx\left(\frac{dy}{dx}\right) = \left(g(x)\,h(y)\right)dx}{h(y)}$

$\frac{dy}{h(y)} = g(x)dx$

2. Integrate and solve for y.

$\boxed{\int \frac{dy}{h(y)} = \int g(x)dx}$

$M(x)dx + N(y)dy = 0$

$N(y)\,dy = -M(x)dx$

$\boxed{\int N(y)dy = -\int M(x)dx.}$

Example 2. Solve the equations/initial value problems:

1. $xydx + (x+1)dy = 0$

$x \neq -1$

$(x+1)dy = -xy\,dx$

$y(x+1)$

$$\int \frac{dy}{y} = -\int \frac{x+1-1}{x+1}dx = -\int \left[\frac{x+1}{x+1} - \frac{1}{x+1}\right]dx$$

$$\ln|y| = -x + \ln|x+1| + C$$

solve for y: $y = e^{-x+\ln|x+1|+C}$
$= e^{-x} \cdot e^{\ln|x+1|} \cdot e^{C}$

$y=0$

$\boxed{y = (x+1)e^{-x} \cdot C_1}$, where $C_1 = e^C$
general solution.

YES $y=0$ might be a solution
$dy = 0$
Plug $y=0$ and $dy=0$ into the equation:
$x(0)dx + (x+1)(0) = 0$

2. $(x^2-1)y' + 2xy^2 = 0, \quad y(0) = 1$

$$dx\left(\frac{dy}{dx}\right) = y' = \left(-\frac{2xy^2}{x^2-1}\right)dx$$

y^2

$$\int \frac{dy}{y^2} = -\int \frac{2x}{x^2-1}dx$$

$$-\frac{1}{y} = -\ln|x^2-1| + C$$

$$y = \frac{1}{\ln|x^2-1| - C} \quad \text{general solution}$$

Find C such that $y(0) = 1$

$$1 = y(0) = \frac{1}{\ln|0-1| - C}$$

$$1 = -\frac{1}{C}, \text{ or } C = -1$$

The solution to IVP: $\boxed{y = \frac{1}{\ln|x^2-1|+1}}$

3. $xydx - \sqrt{x^2+1}\ln^2(y)dy = 0$ $\quad y > 0$

$$\frac{\sqrt{x^2+1}\,\ln^2 y\,dy = xy\,dx}{\sqrt{x^2+1}\,y}$$

$$\int \frac{\ln^2 y}{y}\,dy = \int \frac{x}{\sqrt{x^2+1}}\,dx$$

$$\frac{\ln^3 y}{3} = \frac{1}{2}\,\frac{(x^2+1)^{1/2}}{1/2} + c$$

$$\boxed{\frac{\ln^3 y}{3} = (x^2+1)^{1/2} + c}$$

implicit solution

4. $x\cos^2 y dx - e^x \sin 2y dy = 0, \quad y(0) = 0$

$$\frac{\sin 2y}{\cos^2 y}\,dy = \frac{x}{e^x}\,dx$$

$$\frac{2\sin y\,\cancel{\cos y}}{\cos^{\cancel{2}} y}\,dy = xe^{-x}\,dx$$

$$\int 2\tan y\,dy = \int xe^{-x}\,dx$$

by parts: $\int u\,dv = uv - \int v\,du$

$u = x \quad dv = e^{-x}dx$

$du = dx \quad v = -e^{-x}$

$$-2\ln|\cos y| = -xe^{-x} - \int(-e^{-x})dx$$

$$-2\ln|\cos y| = -xe^{-x} - e^{-x} + C$$

Find C such that $y(0)=0$

Plug $x=0$ and $y=0$ into the equation:

$$-2\ln|\cancel{\cos 0}^{\,1}| = -0e^{-0} - e^{-0} + C$$

$$0 = -1 + C \quad \text{or} \quad C = 1$$

solution to IVP: $\boxed{-2\ln|\cos y| = -xe^{-x} - e^{-x} + 1}$