Section 2.4 **Differences between linear and nonlinear equations.**

- **Linear equations.**

Theorem 1. Suppose $p(t)$ and $q(t)$ are continuous on some interval I that contains the point t_0. Then for any choice of initial value y_0, there exists a unique solution $y(t)$ on I to the initial value problem

p(t), q(t) are continuous on I

$$y' + p(t)y = q(t), \qquad y(t_0) = y_0$$

t_0 I

Example 1. Determine (without solving the problem) an interval in which the solution of the given IVP is certain to exist:

$$(t-3)y' + (\ln t)y = 2t$$

1. $y(1) = 2$

$\boxed{(0,3)}$

standart form:

$$y' + \frac{\ln t}{t-3}y = \frac{2t}{t-3}$$

$$p(t) = \frac{\ln t}{t-3}, \quad q(t) = \frac{2t}{t-3}$$

$p(t)$ is continuous on $(0,3)\cup(3,\infty)$

$q(t)$ is continuous on $(-\infty,3)\cup(3,\infty)$

both $p(t)$ and $q(t)$ are continuous on $(0,3)\cup(3,\infty)$

0 1 3 5

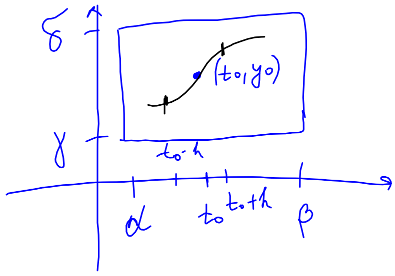
2. $y(5) = 6$

$\boxed{(3,\infty)}$

- **Nonlinear equations.**

Theorem 2. Let the functions f and $\dfrac{\partial f}{\partial y}$ are continuous in some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing the point (t_0, y_0). Then, in some interval $t_0 - h < t < t_0 + h$ contained $\alpha < t < \beta$, there is a unique solution of the initial value problem

$$y' = f(t, y), \quad y(t_0) = y_0$$



Theorem 2 $\Rightarrow$ Theorem 1.

linear equation

$$\frac{dy}{dt} + p(t)y = g(t)$$

rewrite as

$$\frac{dy}{dt} = g(t) - p(t)y = f(t,y)$$

$$\frac{\partial f}{\partial y} = -p(t)$$

Remarks:

1. By this theorem we can guarantee the existence of solution only for values of t which are suffiently closed to t_0, but not for all t.
2. Geometric consequence of the theorem is that two integral curves never intersect each other.
3. The condition "$\dfrac{\partial f}{\partial y}$ is continuous in some rectangle..." is important for uniqueness.

Example 2. Solve the initial value problem

$$y' = y^{1/3}, \; y(0) = 0$$

$y(t) = 0$ is a solution to the initial value problem.

$$\frac{dy}{dt} = y^{1/3} \quad \text{(separable)}$$

$$\int \frac{dy}{y^{1/3}} = \int dt$$

$$\frac{3}{2} y^{2/3} = t + C$$

$$y^{2/3} = \frac{2}{3}(t + C)$$

$$y = \left[\frac{2}{3}(t + C)\right]^{3/2}$$

$$0 = y(0) = \left[\frac{2}{3} C\right]^{3/2} \text{ or } C = 0$$

$y = \left[\frac{2}{3} t\right]^{3/2}$ another solution to the initial value problem.

$f(t,y) = y^{1/3}$ – continuous for all y.

$\dfrac{\partial f}{\partial y} = \dfrac{1}{3} y^{-2/3}$ – not continuous at $y = 0$.

Example 3. For the IVP

$$y' = \frac{\ln|ty|}{1-t^2+y^2}$$

state where in the ty-plane the hypotheses of Theorem 2 are satisfied.

$$y' = \frac{\ln|ty|}{1-t^2+y^2}$$

$$f(t,y) = \frac{\ln|ty|}{1-t^2+y^2} \qquad \text{continuous if } ty \neq 0 \text{ and } 1-t^2+y^2 \neq 0$$

$$\frac{\partial f}{\partial y} = \frac{\frac{1}{y}(1-t^2+y^2) - 2y\ln|ty|}{(1-t^2+y^2)^2}$$

$$= \frac{(1-t^2+y^2) - 2y^2\ln|ty|}{y(1-t^2-y^2)^2} \qquad \text{continuous if } ty \neq 0,\ y \neq 0,\ 1-t^2+y^2 \neq 0$$

$t=0$ - y-axis

$y=0$ - t-axis

$t^2-y^2=1$ hyperbola

$ty \neq 0$

either $t \neq 0$ or $y \neq 0$

$1-t^2+y^2 \neq 0$

$t^2-y^2 \neq 1$