Section 2.6 **Exact equations and integrating factors.**

Given an equation

$$M(x, y)dx + N(x, y)dy = 0.$$

There exists an implicit solution of the equation $F(x, y) = C$ if and only if

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.}$$

$F(x, y)$ satisfies the following conditions:

$$\frac{\partial F(x, y)}{\partial x} = M(x, y), \quad \frac{\partial F(x, y)}{\partial y} = N(x, y)$$

In such a case, the equation is called exact.

Example 1. Is the equation

$$\underbrace{(3x^2 - 2xy + 2)}_{M(x,y)}dx + \underbrace{(6y^2 - x^2 + 3)}_{N(x,y)}dy = 0$$

exact? If it is, solve it.

Check $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

$$\frac{\partial M}{\partial y} = -2x = \frac{\partial N}{\partial x} = -2x \qquad \boxed{\text{EXACT}}$$

Find $F(x,y)$ such that $\begin{cases} \dfrac{\partial F}{\partial x} = M(x,y) \\ \dfrac{\partial F}{\partial y} = N(x,y) \end{cases}$

$$\begin{cases} \dfrac{\partial F}{\partial x} = 3x^2 - 2xy + 2 \\ \displaystyle\int \frac{\partial F}{\partial y}\,dy = \int 6y^2 - x^2 + 3\,dy \end{cases}$$

$F(x,y) = 2y^3 - x^2y + 3y + h(x)$

$h(x)$ is an unknown function

$$\frac{\partial F}{\partial x} = -2xy + h'(x)$$

$$-\cancel{2x}y + h'(x) = 3x^2 - \cancel{2x}y + 2$$

$$h'(x) = 3x^2 + 2$$

$$h(x) = x^3 + 2x + C$$

Update $F(x,y) = 2y^3 - x^2y + 3y + x^3 + 2x + C$

General solution: $\boxed{2y^3 - x^2y + 3y + x^3 + 2x + C = 0}$

Example 2. Solve IVP:

$$3x^2 - y + (2y - x)y' = 0, \quad y(1) = 3$$

$y' = \frac{dy}{dx}$

$(2y-x)dy + (3x^2-y)dx = 0$

$M(x,y) = 3x^2 - y \qquad \frac{\partial M}{\partial y} = -1$

$N(x,y) = 2y - x \qquad \frac{\partial N}{\partial x} = -1$

$\boxed{\text{Exact}}$

$F(x,y): \begin{cases} \int \frac{\partial F}{\partial x} dx = \int 3x^2 - y \, dx \\ \frac{\partial F}{\partial y} = 2y - x \end{cases}$

$F(x,y) = x^3 - xy + h(y)$, $h(y)$ is an unknown function.

$\frac{\partial F}{\partial y} = -x + h'(y)$

$\cancel{-x} + h'(y) = 2y \cancel{-x}$

$h(y) = y^2 + C$

$F(x,y) = x^3 - xy + y^2 + C$

General solution of the equation : $\boxed{x^3 - xy + y^2 + C = 0}$

IC: $y(1) = 3$.

Plug $x=1$ and $y=3$ into the general solution:

$1 - 3 + 9 + C = 0$

$C = -7$

Solution of IVP: $\boxed{x^3 - xy + y^2 - 7 = 0}$

Example 3. Is the equation

$$x^2y^3 + x(1+y^2)y' = 0 \qquad y' = \frac{dy}{dx}$$

exact?

$$x^2y^3\,dx + x(1+y^2)\,dy = 0$$

$$M(x,y) = x^2y^3 \qquad \frac{\partial M}{\partial y} = 3x^2y^2$$

$$N(x,y) = x(1+y^2) \qquad \frac{\partial N}{\partial x} = 1+y^2$$

$\neq$ not exact

Multiply the equation by the integrating factor $\mu(x,y) = \dfrac{1}{xy^3}$ and then solve it.

$$\frac{1}{xy^3}\left[x^2y^3dx + x(1+y^2)dy\right] = 0$$

$$x\,dx + \frac{1+y^2}{y^3}\,dy = 0 \qquad \boxed{\text{Exact}}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 0$$

$$M(x,y) = x$$

$$N(x,y) = \frac{1+y^2}{y^3}$$

$$F(x,y):\ \begin{cases} \displaystyle\int \frac{\partial F}{\partial x}dx = \int x\,dx \\ \displaystyle\frac{\partial F}{\partial y} = \frac{1+y^2}{y^3} \end{cases}$$

$$F(x,y) = \frac{x^2}{2} + h(y)$$

$$\frac{\partial F}{\partial y} = h'(y)$$

$$\int h'(y)\,dy \quad \frac{1+y^2}{y^3}$$

$$= \int y^{-3} + y^{-1}\,dy$$

$$h(y) = \frac{y^{-2}}{-2} + \ln|y| + C$$

$$\boxed{F(x,y) = \frac{x^2}{2} - \frac{y^{-2}}{2} + \ln|y| + C}$$

General solution of the equation:

$$\boxed{\frac{x^2}{2} - \frac{y^{-2}}{2} + \ln|y| + C = 0}$$

Check whether $y=0$ is a solution of

$$x^2y^3 + x(1+y^2)y' = 0$$

$y=0$, then $y'=0$

$$x^2(0) + x(1+0)(0) = 0$$

$$0 = 0$$

$\boxed{y=0 \text{ is a solution.}}$

Given an equation

$$M(x,y)dx + N(x,y)dy = 0.$$

If $\dfrac{M_y - N_x}{N}$ is a function of x only, then a solution $\mu(x)$ of the equation

$$\boxed{\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu}$$

is an integrating factor for the differential equation.

$\dfrac{N_x - M_y}{M}$ is a function on y only

The integration factor $\mu(y)$ is a solution of

$$\boxed{\frac{d\mu}{dy} = \frac{N_x - M_y}{M}\mu}$$

Example 4. Find an integrating factor for the equation

$$(3x^2y + 2xy + y^3)dx + (x^2 + y^2)dy = 0$$

and then solve the equation.

$M(x,y) = 3x^2y + 2xy + y^3$

$N(x,y) = x^2 + y^2$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N(x,y)} = \frac{3x^2 + 2x + 3y^2 - 2x}{x^2+y^2}$$

$= 3$ (does not depend on y)

Integrating factor $\mu(x)$:

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu$$

$$\frac{d\mu}{dx} = 3\mu$$

$$\int \frac{d\mu}{\mu} = \int 3dx$$

$$\ln|\mu| = 3x$$

$$\boxed{\mu(x) = e^{3x}}$$

$$e^{3x}\left[(3x^2y + 2xy + y^3)dx + (x^2+y^2)dy\right] = 0$$

Exact $-$ $(3x^2y + 2xy + y^3)e^{3x}dx + (x^2+y^2)e^{3x}dy = 0$

$M(x,y) = (3x^2y + 2xy + y^3)e^{3x}$ | $\frac{\partial M}{\partial y} = (3x^2 + 2x + 3y^2)e^{3x}$

$N(x,y) = (x^2+y^2)e^{3x}$ | $\frac{\partial N}{\partial x} = 2xe^{3x} + 3(x^2+y^2)e^{3x}$

$F(x,y)$: $\begin{cases} \frac{\partial F}{\partial x} = (3x^2y + 2xy + y^3)e^{3x} \\ \int \frac{\partial F}{\partial y}dy = \int (x^2+y^2)e^{3x}dy \end{cases}$

$$F(x,y) = \left(x^2y + \frac{y^3}{3}\right)e^{3x} + h(x)$$

$$\frac{\partial F}{\partial x} = 2xye^{3x} + 3\left(x^2y + \frac{y^3}{3}\right)e^{3x} + h'(x)$$

$$\cancel{2xy}e^{3x} + 3\left(\cancel{x^2y} + \cancel{\frac{y^3}{3}}\right)e^{3x} + h'(x) = (\cancel{3x^2y} + \cancel{2xy} + \cancel{y^3})e^{3x}$$

$$h'(x) = 0$$

$$h(x) = C$$

$$F(x,y) = \left(x^2y + \frac{y^3}{3}\right)e^{3x} + C$$

General solution of the equation:

$$\boxed{\left(x^2y + \frac{y^3}{3}\right)e^{3x} + C = 0}$$