Section 3.2 **Solutions of linear homogeneous equations; the Wronskian.**

A **linear second order equation** is an equation that can be written in the form

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t). \tag{1}$$

Associated homogeneous equation for (1) is

$$y'' + p(t)y' + q(t)y = 0, \tag{2}$$

Let's consider the expression on the left-hand side of equation (2),

$$y''(t) + p(t)y'(t) + q(t)y(t). \tag{3}$$

Given any function y with a continuous second derivative on the interval I, then (3) generates a new function

$$L[y] = y''(t) + p(t)y'(t) + q(t)y(t). \tag{4}$$

What we have done is to associate with each function y the function $L[y]$. This function L is defined *on a set of functions*. Its domain is the collection of functions with continuous second derivatives; its range consists of continuous functions; and the rule of correspondence is given by (4). We will call this mappings **operators**. Because L involves differentiation, we refer to L as a **differential operator**.

The image of a function y under the operator L is the function $L[y]$. If we want to evaluate this image function at some point t, we write $L[y](t)$.

Example 1. Let $L[y](t) = t^2y''(t) - 3ty'(t) - 5y(t)$. Compute

1. $L[\cos t]$

2. $L[t^{-1}]$;

3. $L[e^{rt}]$, r a constant.

There are *basic differentiation operators* with respect to t:

$$Dy = \frac{dy}{dt}, \quad D^2y = \frac{d^2y}{dt^2}, \dots, D^ny = \frac{d^ny}{dt^n}.$$

Using these operators we can express L defined in (4) as

$$L[y] = D^2y + pDy + qy = (D^2 + pD + q)y.$$

When p and q are *constants*, we can even treat $D^2 + pD + q$ as a polynomial in D and factor it.

The differential operator L defined by (4) has two very important properties.

Lemma. Let $L[t] = y''(t) + p(t)y'(t) + q(t)y(t)$. If y, y_1, and y_2 are any twice-differentiable functions on the interval I and if c is any constant, then

$$L[y_1 + y_2] = L[y_1] + L[y_2], \tag{5}$$

$$L[cy] = cL[y]. \tag{6}$$

Any operator that satisfied satisfies properties (5) and (6) for any constant c and any functions y, y_1, and y_2 in its domain is called a **linear operator** and we can say that *"L preserves linear combination"*. If (5) or (6) fails to hold, the operator is **nonlinear**.

Lemma says that the operator L, defined by (4) is linear.

Theorem 1 (Principle of superposition). Let y_1 and y_2 be solutions to the *homogeneous* equation (2). Then any linear combination $C_1y_1 + C_2y_2$ of y_1 and y_2, where C_1 and C_2 are constants, is also the solution to (2).

Example 2. Verify that $y_1(t) = 1$ and $y_2(t) = t^{1/2}$ are solutions of the differential equation $yy'' + (y')^2 = 0$ for $t > 0$. Then show that $y = c_1 + c_2t^{1/2}$ is not, in general, a solution of this equation. Explain why this result does not contradict Theorem 1.

Theorem 2 (existence and uniqueness of solution). Suppose $p(t)$, $q(t)$, and $g(t)$ are continuous on some interval (a, b) that contains the point t_0. Then, for any choice of initial values y_0, y_1 there exists a unique solution $y(t)$ on the whole interval (a, b) to the initial value problem

$$y'' + p(t)y' + q(t)y = g(t),$$
$$y(t_0) = y_0, \quad y'(0) = y_1.$$

Example 3. Find the largest interval for which Theorem 2 ensures the existence and uniqueness of solution to the initial value problem

$$e^t y'' - \frac{y'}{t-3} + y = \ln t,$$

$$y(1) = y_0, \qquad y'(1) = y_1,$$

where y_0 and y_1 are real constants.

Fundamental solutions of homogeneous equations

Theorem 3. Let y_1 and y_2 denote two solutions on I to

$$y'' + p(t)y' + q(t)y = 0,$$

where $p(t)$ and $q(t)$ are continuous on I. Suppose at some point $t_0 \in I$ these solutions satisfy

$$y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0) \neq 0. \tag{7}$$

Then every solution to (2) on I can be expressed in the form

$$y(t) = C_1 y_1(t) + C_2 y_2(t), \tag{8}$$

where C_1 and C_2 are constants.

Definition For any two differentiable functions y_1 and y_2, the determinant

$$W[y_1, y_2](t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_1'(t)y_2(t)$$

is called the **Wronskian** of y_1 and y_2.

Example 4. Find the Wronskian for the functions $e^t \sin t$, $e^t \cos t$.

Example 5. If the Wronskian of f and g is $3e^{4t}$, and if $f(t) = e^{2t}$, find $g(t)$.

Definition 2. A pair of solutions $\{y_1, y_2\}$ to $y'' + p(t)y' + q(t)y = 0$ on I is called **fundamental solution set** if

$$W[y_1, y_2](t_0) \neq 0$$

at some $t_0 \in I$.